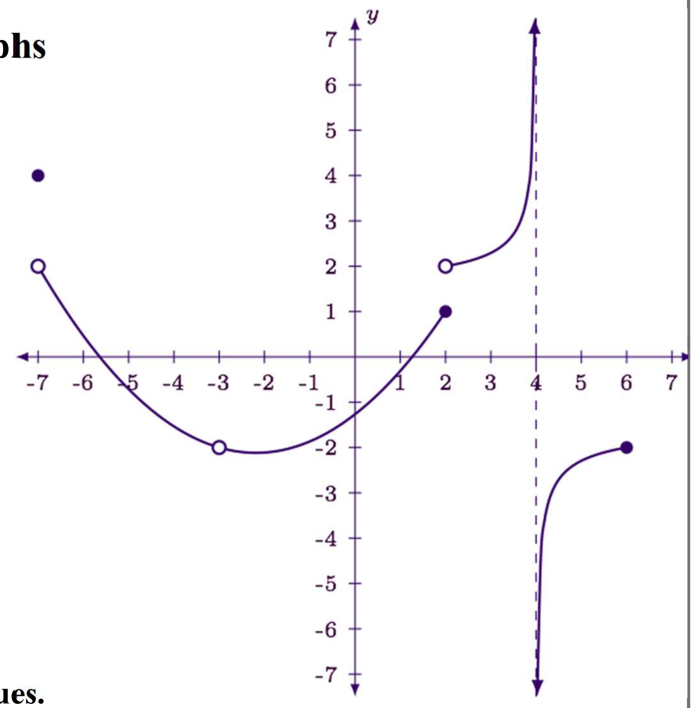
11.1 Notes: Finding Limits Using Tables and Graphs



Functional Notation:

Types of Discontinuities:

Examples: Use the graph of $f(x)$ to find the requested values.

1) $f(2)$

2) $f(-2)$

3) $f(-7)$

4) $f(-3)$

5) $f(6)$

6) $f(4)$

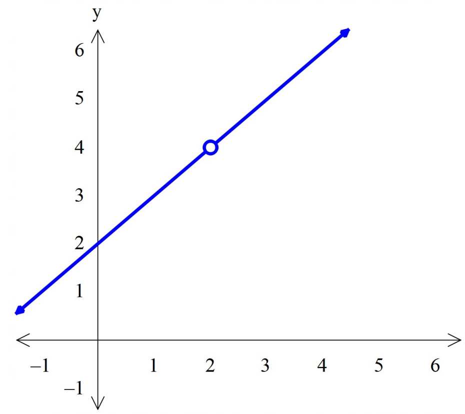
7) Identify all x-values where $f(x)$ has a discontinuity and yet is still defined.

Also, what type(s) of discontinuities exist at those x-values?

8) Identify all x-values where $f(x)$ has a discontinuity and is undefined.

Also, what type of discontinuity exists at those x-values?

Example 10: Consider the graph shown of $g(x) = \frac{x^2-4}{x-2}$.



- What height does $g(x)$ approach as x approaches 2 from both the left and right sides?

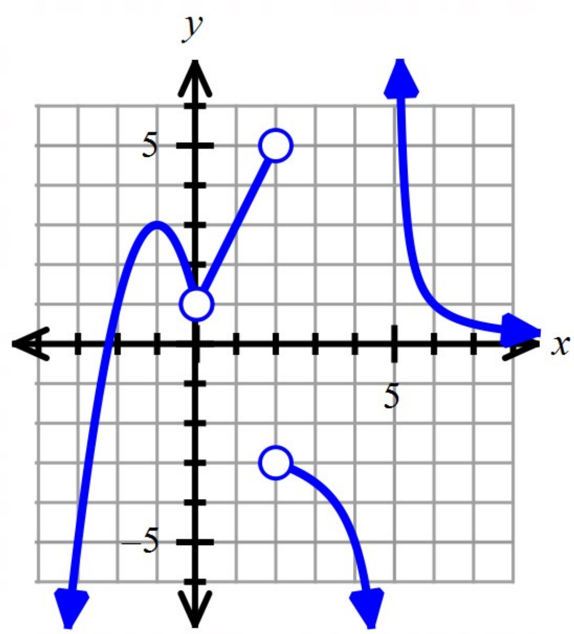
- Describe the graph of $g(x)$ as $x = 2$.

- Fill out the table of values below by using your calculator.

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$g(x)$							

Definition of a Limit	Limit Notation
	$\lim_{x \to a} f(x) = L$

Examples: Use the graph of $f(x)$ to find the requested limits.



11) $\lim_{x \to 1} f(x)$

12) $\lim_{x \to 0} f(x)$

13) $\lim_{x \to 4} f(x)$

14) $\lim_{x \to 2} f(x)$

15) $\lim_{x \to 5} f(x)$

16) $\lim_{x \to -1} f(x)$

17) Identify all x-values where $f(x)$ has a discontinuity and yet is the limit still exists.

Also, what type(s) of discontinuities exist at those x-values?

18) Identify all x-values where $f(x)$ has a discontinuity and the limit does NOT exist.

Also, what type(s) of discontinuities exist at those x-values?

For Examples 19 – 20, use a table of values to find the following limits. Verify your conclusion by graphing each function on your calculator.

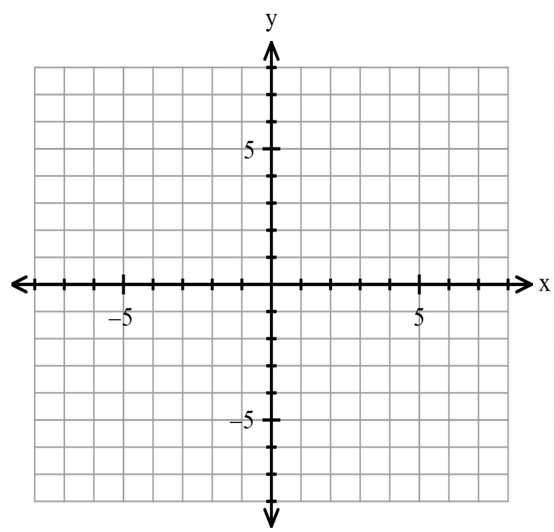
19) $\lim_{x \to 4} 3x^2$

x							
$3x^2$							

20) $\lim_{x \to 0} \frac{\sin}{x}$

x							
$\frac{\sin x}{x}$							

For #21 – 24, use the graph of $h(x) = \begin{cases} \frac{2}{3}x - 1 & if\ x \neq 3 \\ -5 & if\ x = 3 \end{cases}$.



21 Find $h(0)$.

22) Find $h(3)$.

23) Find $\lim_{x \to 0} h(x)$

24) Find $\lim_{x \to 3} h(x)$

One-Sided Limits	One-Sided Limit Notation
	$\lim_{x \to a^+} f(x) = L$ $\lim_{x \to a^-} f(x) = L$

One-Sided Limits and the Existence of a Two-Sided Limit

For #25 – 30, use the graph from the #21 – 24 of $h(x)$.

25) Find $\lim_{x \to 1^+} h(x)$

26) Find $\lim_{x \to 1^-} h(x)$

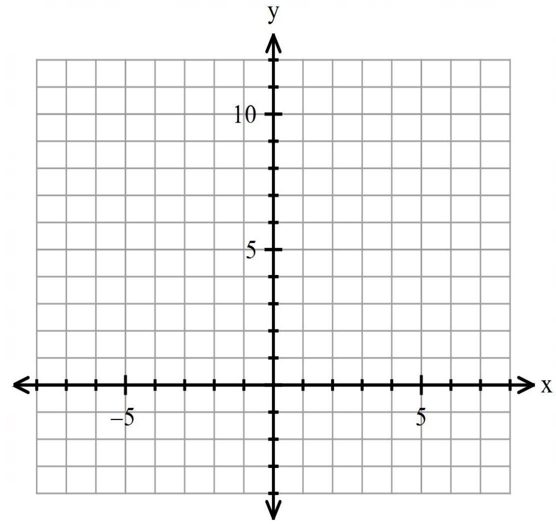
27) Find $\lim_{x \to 1} h(x)$

28) Find $\lim_{x \to 3^+} h(x)$

29) Find $\lim_{x \to 3^-} h(x)$

30) Find $\lim_{x \to 3} h(x)$

For #31 – 34, use the graph of $f(x) = \begin{cases} x^2 - 2 & \text{if } x < 1 \\ x + 3 & \text{if } x \geq 1 \end{cases}$



Find each value.

31) $\lim_{x \to 1^-} f(x)$

32) $\lim_{x \to 1^+} f(x)$

33) $\lim_{x \to 1} f(x)$

34) $f(1)$

For #35 – 38, determine if each statement is true or false. If it is false, sketch a counter-example. Note: c **is some constant.**

35) If $f(x)$ is continuous at $x = c$, then $\lim_{x \to c} f(x)$ exists.

36) If $f(c)$ is defined, then $\lim_{x \to c} f(x)$ exists.

37) If $f(x)$ has a discontinuity at $x = c$, then $\lim_{x \to c} f(x)$ does not exist.

11.2 Notes: Finding Limits with Analytical Methods

What are Analytical Methods?

Properties of Limits:

- **Sum Property:** If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}[f(x) + g(x)] =$

  $\lim_{x\to a} f(x) + \lim_{x\to a} g(x) =$ _______________

- **Difference Property:** If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

  $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) =$ _________________

- **Product Property:** If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

  $\lim_{x\to a}[f(x) * g(x)] = \lim_{x\to a} f(x) * \lim_{x\to a} g(x) =$ ___________________

- **Limit of a Quotient:** If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}, M \neq 0$

For Examples 1 – 4, given that $\lim_{x\to 3} f(x) = -8$ and $\lim_{x\to 3} g(x) = 5$, then find each requested limit below.

1) $\lim_{x\to 3}(f(x) + g(x))$

2) $\lim_{x\to 3}(f(x) \cdot g(x))$

3) $\lim_{x\to 3} \frac{f(x)}{g(x)}$

4) $\lim_{x\to 3}(4f(x) - 2g(x))$

Using Substitution to Evaluate Limits:

For #5 – 15: Find the following limits.

5) $\lim_{x \to -4} (x + 9)$

6) $\lim_{x \to 5} (12 - x)$

7) $\lim_{x \to 5} (-6x)$

8) $\lim_{x \to -3} (7x - 4)$

9) $\lim_{x \to 5} -6x^3$

10) $\lim_{x \to 3} (4x^3 + 2x^2 - 6x + 5)$

11) $\lim_{x \to 5} (2x - 7)^3$

12) $\lim_{x \to -2} \sqrt{4x^2 + 5}$

13) $\lim_{x \to 1} \frac{x^3 - 3x^2 + 7}{2x - 5}$

14) $\lim_{x \to 4} 3$

15) $\lim_{x \to -\pi} x$

16) Given that $f(x) = \begin{cases} x^2 + 5 & if\ x < 2 \\ 3x + 1 & if\ x \geq 2 \end{cases}$, find each of the following limits:

a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

c. $\lim_{x \to 2} f(x)$

Evaluating Limits at Points of Discontinuity:

- Reminder: Limits can exists at some discontinuities (holes) in the graph.
- When substitution results in $\frac{0}{0}$, this indicates a hole in the graph.
- Re-write the expression with one of the methods below to evaluate the limit.
    1. Factor and reduce
    2. Rationalize the numerator

For #17 – 20, find each limit.

17) $\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3}$

18) $\lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x}$

19) $\lim_{x \to 0} \frac{7x^2 - 3x}{x}$

20) $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$

For #21: Given an expression for $f(x)$, find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

21) $f(x) = -2x^2 + 5$

For #22 – 24: Given an expression for $f(x)$, find $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

22) $f(x) = \sqrt{x}$

23) $f(x) = 2x^2 - 4x$

24) $f(x) = \sqrt{x+5}$

11.3 Notes: Limits and Continuity

Continuous Functions: What does a continuous function look like?

- What types of functions are continuous?

- How can we identify discontinuities by examining an equation for a function?

- **Reminder: Types of discontinuities:**
  1) Holes
  2) Jump Discontinuities
  3) Vertical Asymptotes

For Examples 1 – 2, is $f(x)$ continuous at the given values of x? If not, describe the type of discontinuity.

1) $f(x) = \frac{2x+1}{2x^2-x-1}$

a) at $x = 2$ b) at $x = 1$ c) $x = \frac{1}{2}$

2) Determine if $f(x) = \frac{x-2}{x^2-4}$ is continuous

a) at $x = 1$ b) at $x = 2$

Definition of Continuous	$f(x)$ is continuous at $x = a$ when $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

For #3 – 5: For each piecewise function given, find values of x that are discontinuities, if any. Also, describe the type of discontinuity.

3) $f(x) = \begin{cases} x+2 & \text{if } x \leq 0 \\ 2 & \text{if } 0 < x \leq 1 \\ x^2+2 & \text{if } x > 1 \end{cases}$

4) $g(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ x^2+1 & \text{if } 0 < x \leq 2 \\ 7-x & \text{if } x > 2 \end{cases}$

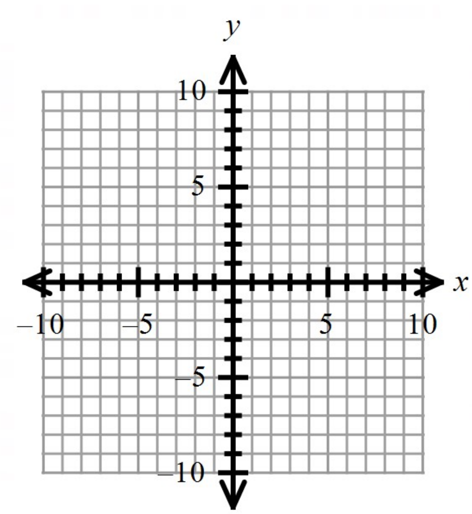
5) $h(x) = \begin{cases} \frac{x^2+x-6}{x^2+4x+3} & \text{if } x \neq -3 \\ 2.5 & \text{if } x = 3 \end{cases}$

For #6 – 8: Given that $f(x) = \begin{cases} 3x - 1 \text{ if } x < 2 \\ -2x + 9 \text{ if } x > 2 \end{cases}$

6) Find, if it exists, $\lim_{x \to 2} f(x)$. (Hint: consider one-sided limits)

7) Is f continuous at $x = 2$? Use the definition of continuity to justify your conclusion.

8) Draw a sketch of $f(x)$ on the provided coordinate system. Compare your graph with your conclusions from #6 and #7, and adjust your conclusions, as needed.

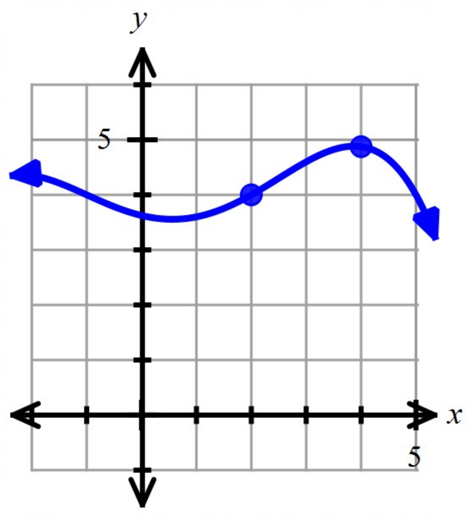


For #9 – 10: True or False? If false, then sketch a counter-example.

9) If $g(x)$ is continuous at $x = a$ then $\lim_{x \to a} g(x)$ exists.

10) If $\lim_{x \to a} g(x)$ exists, then $g(x)$ is continuous at $x = a$.

11.4 Notes: Introduction to Derivatives



Consider the graph of $f(x)$ as shown.

Suppose you wanted to find the rate of change (or "slope") of $f(x)$ at various locations of x.

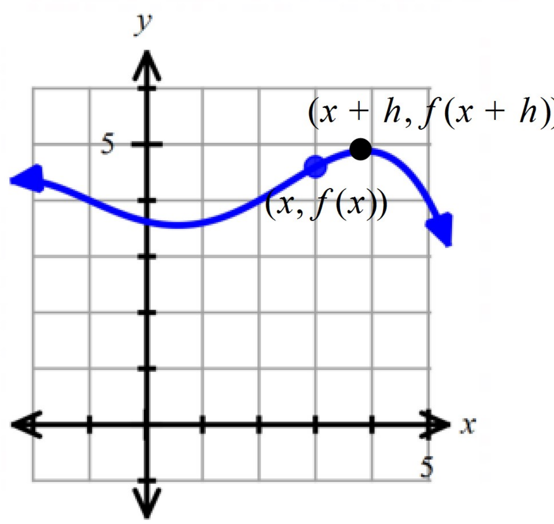
- Find the rate of change by calculating the slope between the two indicated points on the graph. This is called the slope of the _______________ line.

- Assume that you were primarily interested in the rate of change of $f(x)$ at the instance when $x = 3$. How can we find this slope? Note: we can call this the slope of the _____________________ line.

  - The slope of the secant line that we found earlier can be used to _________________ the slope of the tangent line at $x = 3$. How could we make this approximation closer to the actual slope at $x = 3$ (or at any other singular point)?

Using the points $(x, f(x))$ and $(x + h), f(x + h)$, Calculate the slope of a generalized secant line.

Slope $= \frac{\text{rise}}{\text{run}} =$



Use limits to move the two points closer together, until they are at the same location.

- This is one of the main areas of interest in the study of calculus: the rate of change of a function at *one* point, also called the *instantaneous rate of change* or the _____________________ of a function.

Limit Definition of a Derivative	$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	**Equivalent Phrases for a Derivative**	• Slope of tangent line • Instantaneous rate of change • Instantaneous velocity
Notation for a Derivative		**Finding the slope of a tangent line at a point**	
Writing the equation of a tangent line in (h, k) form			

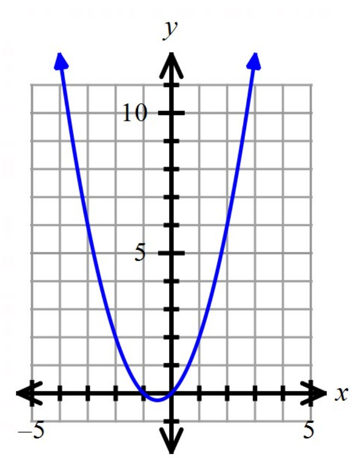
For #1 – 2, use the limit definition of a derivative to find $g'(x)$.

1) $g(x) = 3x - 2$

2) $g(x) = 2x^2 - 1$

For #3 – 6: Use $f(x) = x^2 + x$.

3) Find the derivative of f. In other words, find $f'(x)$ by using the limit definition of a derivative.



4) Find the slope of the tangent line to $f(x)$ at $(2, 6)$.

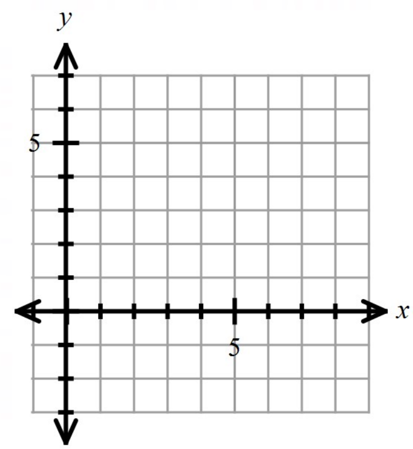
5) Write the equation of the tangent line, in (h, k) form.

6) Sketch the tangent line you found in #5. Note: $f(x)$ is already graphed for you.

For #7 – 10: Use $h(x) = \sqrt{x}$.

7) Find $h'(x)$ by using the limit definition of a derivative.

8) Find the slope of the tangent line to $h(x)$ at $(4, 2)$.



9) Write the equation of the tangent line, in (h, k) form.

10) Sketch $f(x)$ and the tangent line you found in #9.

For Examples 11 – 12: A ball is thrown straight up from a rooftop 160 ft high with an initial velocity of 48 feet per second. The function $s(t) = -16t^2 + 48t + 160$ describes the ball's height above the ground in feet, t seconds after it is thrown. The ball misses the rooftop on its way down and eventually strikes the ground.

11) Given that $s'(t) = -32t + 48$, what is the instantaneous velocity of the ball 2 seconds after it is thrown?

12) What is the instantaneous velocity of the ball when it hits the ground? (Hint: find t first.) Why is your answer negative?

13) Hazel attempted to find the derivative of $f(x) = -2x^2 + 5x$? Which set-up below is the correct limit to find $f'(x)$?

A) $\lim_{h \to 0} \frac{-2(x+h)^2+5(x+h)-2x^2+5x}{h}$

B) $\lim_{h \to 0} \frac{-2x^2+5x+h-2x^2+5x}{h}$

C) $\lim_{h \to 0} \frac{-2x^2+5x+h+2x^2-5x}{h}$

D) $\lim_{h \to 0} \frac{-2(x+h)^2+5(x+h)+2x^2-5x}{h}$

11.5 Notes: The Power Rule for Derivatives

As mathematicians explored the limit definition of a derivative, they noticed patterns that allowed them to prove various rules that allow us to calculate a derivative with much more ease, the first of which is the Power Rule.

Power Rule	Given that n is a constant: $$\frac{d}{dx}(x^n) = nx^{n-1}$$	Given that n and a are constants: $$\frac{d}{dx}(ax^n) = anx^{n-1}$$
Addition and Subtraction with the Power Rule	$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$	
Derivative of a Constant	Given that n is a constant: $$\frac{d}{dx}(n) = \_\_\_\_\_\_$$	

Examples 1 – 6: Find the derivative of each expression.

1) $y = x^5$

2) $f(x) = -8y^3$

3) $h(x) = -7$

4) $g(x) = -2x^5 + 9x^2 + 4x - 3$

5) $y = 4x^{\frac{3}{2}} + 9$

6) $b(x) = -5x^{-\frac{1}{2}}$

Hints for using the Power Rule	**As needed, before taking the derivative, re-write expressions so that the value of the exponent can be determined.** Write all *radical* terms with a ______________ exponent. $\sqrt{x} = x^{\frac{1}{2}}$ $\sqrt[3]{y^2} = y^{\frac{2}{3}}$ Move all variables that are on the *denominator* of a rational expression to the numerator by using ___________________ exponents. $\frac{3}{y^5} = 3y^{-5}$ $\frac{1}{7b^2} = \frac{1}{7}b^{-2}$

For #7 – 12, rewrite each expression and then find the derivative.

7) $h(x) = 18\sqrt{x}$

8) $y = \frac{11}{t^5} + t$

9) $f(x) = -\frac{2}{5x^3} - x^{34}$

10) $y = -10\sqrt{x^5}$

11) $g(r) = 9\sqrt[3]{r^2} + 39$

12) $y = \frac{8}{\sqrt{x}}$

Exploration: Consider $g(x) = (5x + 1)(2x - 3)$.

- What do you anticipate $g'(x)$ would be?
- Expand $g(x)$ and then find the derivative of that expression.
- Compare the results with your prediction. What do you notice?

Simplifying Expressions Before Using the Power Rule	**As needed, before taking the derivative, re-write expressions so that the value of the exponent can be determined.** ________________ expressions that show a product. • $(3x-2)^2 = 9x^2 - 12x + 4$ • $4x^3(6x-7) = 24x^4 - 28x^3$ _______________ rational expressions where the denominator is a monomial. • $\frac{x^2-4x^3+2x-3}{x} = x - 4x^2 + 2x - 3x^{-1}$ • $\frac{9x^5-18x^4+2x}{3x^3} = 3x^2 - 6x + \frac{2}{3}x^{-2}$

For #13 – 16, rewrite each expression and then find the derivative.

13) $f(x) = (5x+4)^2$

14) $y = -2x(6x-5)$

15) $y = \frac{7x^5-8x^3+15x-}{x}$

16) $g(x) = \frac{16x^4-12x^2-6x}{4x^2}$

Reminder for writing equations of tangent lines	$\boldsymbol{y = m(x-h)+k}$ To find the slope m: • Find y' • Evaluate the derivative at the given value of x.

17) Given $g(x) = -3x^2 - 7x + 2$, write the equation of the tangent line to $g(x)$, in (h,k) form, at $(2,-24)$.

18) Find the slope of the tangent line to the $y = 2x + \sqrt{x}$ at $x = 36$.

19) Find the ordered pairs on $y = 2x^3 + \frac{13}{2}x^2 + 2x + 4$ where the tangent line would be horizontal.

11.6: The Product Rule

The Product Rule	Given two functions $f(x)$ and $g(x)$. Then $$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

Exploration: Consider $f(x) = 3x - 2$ and $g(x) = -4x + 5$, find $\frac{d}{dx}(f(x)\cdot g(x))$ in two ways:

a) by expanding the product and taking the derivative of the resulting polynomial

b) by using the Product Rule

For Examples 1 – 3, find the derivative of each expression by the method of your choice.

1) $g(x) = -5x^3(7x - 11x^4)$

2) $y = (2x + 3)(x^2 - 6)$

3) $y = \left(\sqrt{x} + x\right)(2x^2 + 1)$

For #4 – 8, use the table of values to find the information, given that $h(x) = f(x) \cdot g(x)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	7	-4	-1	5
3	-3	8	$\frac{1}{2}$	10
−1	9	$\frac{1}{3}$	-12	6

Hint: set-up these problems by first finding $h'(x)$ by using the product rule. Then use the table to evaluate the derivative.

$h'(x) =$

4) Find $h'(-1)$.

5) Find $h'(3)$.

6) Find $h'(2)$.

7) Write the equation of the tangent line, in (h, k) form, to $h(x)$ at $(2, -7)$.

8) Write the equation of the tangent line, in (h, k) form, to $h(x)$ at $(-1, -108)$.

For #9 – 10, find the requested derivative. You do *not* need to simplify your answer.

9) Expand first and then use the power rule to find $f'(x)$ if $f(x) = (\sqrt{x} + 4\sqrt[3]{x})(x^5 - 11x^8)$.

10) Use the product rule to find y' if $y = \left(\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3}\right)\left(\frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5}\right)$.

11) Given that $\lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = -18x^2 + 8x - 12$. Could $g(x) = -2x(3x^2 - 2x + 6)$? Justify your conclusion.

12) Given that the tangent line to $f(x)$ at $(1, 15)$ is $y = 26(x-1) + 15$. Could $f(x) = (4x-1)(2x+3)$? Explain your reasoning.

11.7 Notes: The Quotient Rule

The Quotient Rule	Given two functions $f(x)$ and $g(x)$. Then $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\left(g(x)\right)^2}$$ "*low d high minus high d low over low low*"

For #1 – 4: Find the derivative of each expression. Keep the denominator in factored form (do not expand.)

1) $y = \frac{2x^2+1}{x+5}$

2) $f(x) = \frac{6}{x^3+2}$.

3) $y = \frac{x^2-3}{x-3}$

4) $f(x) = \frac{x^5+3x^4+1}{x^2+7x}$

For #5 – 9, use the table of values to find the information, given that $h(x) = \frac{f(x)}{g(x)}$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	7	-4	-1	5
3	-3	8	$\frac{1}{2}$	10
−1	9	$\frac{1}{3}$	-12	6

Hint: set-up these problems by first finding $h'(x)$ by using the quotient rule. Then use the table to evaluate the derivative.

$h'(x) =$

5) Find $h'(-1)$.

6) Find $h'(3)$.

7) Find $h'(2)$.

8) Write the equation of the tangent line, in (h, k) form, to $h(x)$ at $(2, -7)$.

9) Write the equation of the tangent line, in (h, k) form, to $h(x)$ at $(-1, -108)$.

For #10 – 12: Given that $h(x) = \frac{x^2+8x-1}{4x}$.

10) Find $h'(x)$ by first dividing the expression and using the product rule.

11) Find $h'(x)$ by using the Quotient Rule.

12) Use algebraic simplification to verify that your answers for 10) and 11) are equivalent.

For #13 – 15: Consider $g(x) = \frac{\sqrt{x}+1}{3x^3}$.

13) Re-write $g(x)$ by first dividing the expression, and then find $g'(x)$ by using the Power Rule.

14) Use the Quotient Rule to find $g'(x)$ by using the original expression $g(x) = \frac{\sqrt{x}+1}{3x^3}$.

15) Use algebraic simplification to verify that your answers for 13) and 14) are equivalent.

For #16 - 17, find the derivative of each expression. Hint: you can re-write each expression in an equivalent form.

16) $f(x) = \frac{x^3-\sqrt{x}}{3x}$

17) $y = \frac{x^2+2x}{x+2}$ (hint… you can re-write this expression first!)

11.8 Notes: The Chain Rule

The Chain Rule	Given two functions $f(x)$ and $g(x)$. Then $$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$ In words:

1) Given that $y = 3(5x - 4)^2$. Find y' with the given methods.

a) Expand y and then find y' by using the power rule.

b) Use the chain rule.

c) Use algebra to show that your answers for a) and b) are equivalent.

For #2 – 3: Use the chart below to find the requested value.

	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
x = -2	-1	3	5	-7
x = 3	8	2	-2	4
x = -1	1	11	6	-9

Hint: set-up these problems by first finding each derivative by using the chain rule.

$\frac{d}{dx}[f(g(x))] =$

$\frac{d}{dx}[g(f(x))] =$

2) $\frac{d}{dx}[f(g(-2))]$

3) $\frac{d}{dx}[g(f(-2))]$

For #4 – 8: Find the derivative of each expression. Do not expand your answer.

4) $y = (5x^3 + 4x)^6$

5) $y = 4(3x - 17)^{15}$

6) $h(x) = 7(18x - 4x^2)^{-2}$

7) $f(x) = \sqrt{x^3 + 4x}$

8) $y = \left(2\sqrt{x} + 3x\right)^2$

For #9 – 10: Consider $y = \sqrt{(x^5 - 8x^3)(x^2 + 6x)}$

9) Find y' by first expanding inside the radical, and then using the chain rule.

10) Find y' by using the chain rule and the product rule.

11) Write the equation of the tangent line, in (h, k) form, for $y = \left(2\sqrt{x} + 3x\right)^2$ at $(4, 256)$. (Note: see #8.)

12) **Multiple Choice:** Which option below shows the correct derivative of $y = -\frac{1}{4}(17 - 3x^2)^8$?

A) $y' = -2(-6x)^7$
B) $y' = -2(17 - 3x^2)^7$
C) $y' = -12x(17 - 3x^2)^7$
D) $y' = 12x(17 - 3x^2)^7$