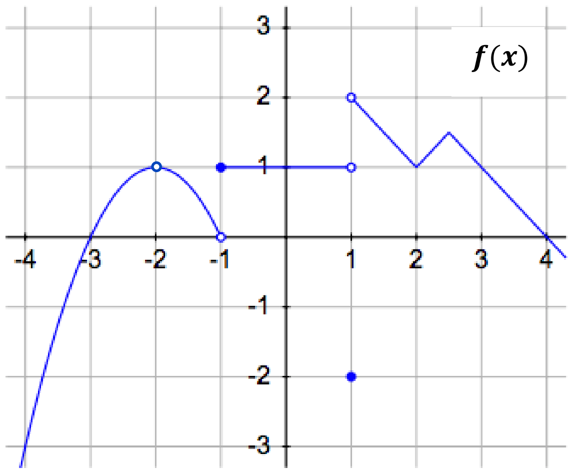
2.2 Notes Introduction to Limits using Graphs and Tables:



Example 1: Use the graph of f(x) above to find the following values:

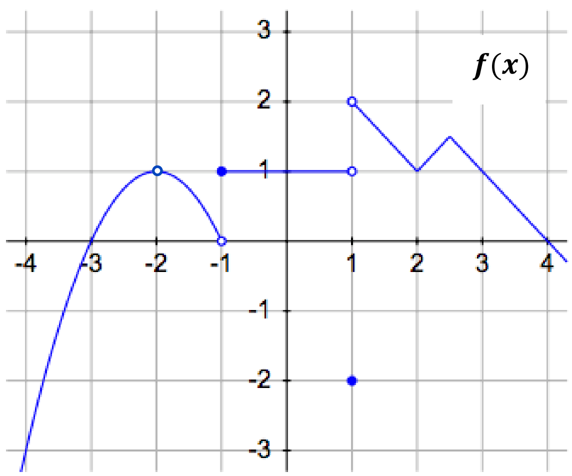
a.) $f(-2)$ b.) $f(-1)$ c.) $f(1)$ d.) $f(4)$

Limit of a Funciton:

Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L for all x sufficiently close (but not equal) to a, we write

$$\lim_{x\to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.



Example 2: Use the graph of f(x) above to find the following values:

a.) $\lim_{x\to -2} f(x) =$ b.) $\lim_{x\to -1} f(x) =$ c.) $\lim_{x\to 1} f(x) =$ d.) $\lim_{x\to 4} f(x) =$

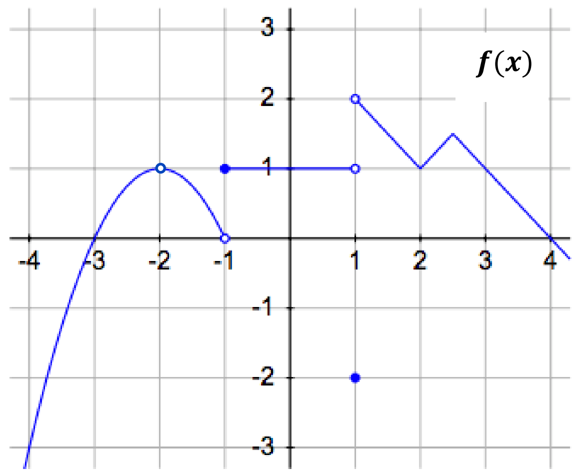
One-Sided Limits:

1. **Right-sided limit:** Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

2. **Left-sided limit:** Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$



Example 3: Use the graph of f(x) above to find the following values:

a.) $\lim_{x \to -2^-} f(x) =$ b.) $\lim_{x \to -1^-} f(x) =$ c.) $\lim_{x \to 1^-} f(x) =$ d.) $\lim_{x \to 4^-} f(x) =$

$\lim_{x \to -2^+} f(x) =$ $\lim_{x \to -1^+} f(x) =$ $\lim_{x \to 1^+} f(x) =$ $\lim_{x \to 4^+} f(x) =$

***Relationship between One-Sided and Two-Sided Limits**

Using tables to approximate limits

Example 4: Use the following table to evaluate $\lim_{x \to 2} g(x)$, where $g(x) = \frac{x-2}{x^2-4}$.

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$g(x)$							

2.2 (cont.) Notes Definitions of Limits

Objective: Students will be able to find a limit of a function, including piecewise functions, using numerical and graphical methods.

Opener: What is a limit? Draw a graph that meets the following requirements:
$\lim_{x \to 0} f(x) = 3$ and $\lim_{x \to 2} f(x) = -1$. What would you expect the equation of this function to be? Is your solution the only correct one?

$$\lim_{x \to c} f(x) = L$$

Example 1: Create a graph for $f(x) = \begin{cases} 4 \ if \ x \neq -1 \\ -3 \ if \ x = -1 \end{cases}$.

- Find f (3).
- Find f (-1).
- Find $\lim_{x \to -1} f(x)$.

Conclusion for functions with one hole:

Example 2: Graph $g(x) = \frac{|x-3|}{x-3}$

- Find g(2).
- Find g (3).
- Find $\lim_{x \to 2} g(x)$.
- Find $\lim_{x \to 3} g(x)$. **(*hint:* use a left and right handed limit)**

Conclusion for behaviors that differ from the left and the right:

Example 3: Graph $h(x) = \frac{4}{x^2}$

- Find $\lim_{x \to 0} h(x)$.
- Find $\lim_{x \to \infty} h(x)$.

Conclusion for unbounded behavior:

Example 4: Find $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$. Use your graphing calculator; x-window to -0.5 $\leq x \leq$ 0.5 with intervals of 0.1.

Conclusion for oscillating behavior:

Why is this a technology pitfall?

Common Types of Functions with Nonexistence of a Limit

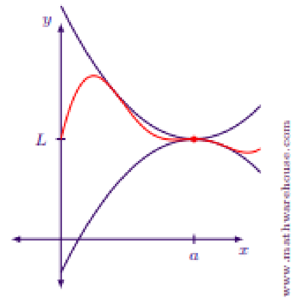
1. $F(x)$ approaches a different value from the right and the left side of c.
2. $F(x)$ increases or decreases without bound as x approaches c.
3. $F(x)$ oscillates between two fixed values as x approaches c.

The Squeeze Theorem (read this on your own)

- If two functions squeeze together at a particular point, then any function trapped between them will get squeezed to that same point.
- The Squeeze Theorem deals with limit values, rather than function values.
- The Squeeze Theorem is sometimes called the *Sandwich Theorem* or the *Pinch Theorem*.

Graphical Example

In the graph shown, the lower and upper functions have the same limit value at $x = a$. The middle function has the same limit value because it is trapped between the two outer functions.



The middle function is squeezed to L as x approaches a.

Definition of the Squeeze Theorem:

Suppose $f(x) \le g(x) \le h(x)$ for all x in an open interval about a (except possibly at a itself). Further, suppose

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Note that the exception mentioned in the statement of the theorem is because we are dealing with limits. That means we're not looking at what happens at $x = a$, just what happens close by.

Example 1

Suppose there are three functions such that $f(x) \le g(x) \le h(x)$ when x is near 2.

Further, suppose $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{7}{3}$ and $h(x) = \cos\left(\frac{\pi}{2}x\right)$ (with x measured in radians).

Determine $\lim_{x\to 2} g(x)$

Solution

Step 1) Find $\lim_{x\to 2} f(x)$.

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2}\left(-\frac{1}{3}x^3 + x^2 - \frac{7}{3}\right)\\ &= \left(-\frac{1}{3}(2)^3 + (2)^2 - \frac{7}{3}\right)\\ &= \left(-\frac{8}{3} + 4 - \frac{7}{3}\right)\\ &= -1\end{aligned}$$

Step 2) Find $\lim_{x\to 2} h(x)$.

$$\begin{aligned}\lim_{x\to 2} h(x) &= \lim_{x\to 2} \cos\left(\frac{\pi}{2}x\right)\\ &= \cos\left(\frac{\pi}{2}(2)\right)\\ &= \cos(\pi)\\ &= -1\end{aligned}$$

Step 3) Conclusion

Since $f(x) \le g(x) \le h(x)$ and $\lim_{x\to 2} f(x) = \lim_{x\to 2} h(x) = -1$, the Squeeze Theorem guarantees $\lim_{x\to 2} g(x) = -1$ as well.

2.3 Notes Techniques for Computing Limits

Objective: Students will be able to evaluate limits analytically.

Continuous Functions are well-behaved and can be evaluated for a limit over their domains by using substitution:

- Constant Functions
- Polynomial Functions
- Rational Functions
- Radical Functions
- Trig Functions
- Composite Functions of other well-behaved functions

Examples: Find each limit, if possible.

1) $\lim_{x\to 16} \frac{12(\sqrt{x}-3)}{x-9}$

2) $\lim_{x\to 3} \sin\frac{\pi x}{2}$

3) $\lim_{x\to -25} \sqrt[3]{x+1}$

Explore: If $f(x) = \frac{x^2-3x+2}{x-1}$, then find $\lim_{x\to 1} f(x)$ in the following manners:

a) substitution

b) graphically

c) analytically

Let c be a real # and $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If $\lim_{x \to c} g(x)$ exists, then $\lim_{x \to c} f(x)$ is the same value.

2 methods for finding the limit analytically when *c* is not in the domain:

1) Rewrite the function so that the undefined value is reduced out.
   a. Factor
   b. Simplify Complex Fractions
2) Rationalize the numerator.

Examples: Find the requested limit analytically,

1) $\lim_{x \to -b} \frac{(x+b)^7 + (x+b)^{10}}{4(x+b)}$

2) $\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$

Example:

Find constants b and c in the polynomial $p(x) = x^2 + bx + c,$ such that $\lim_{x \to 2} \frac{p(x)}{x-2} = 6$. Are the constants unique?

2.3 (Day 2): Special Trig Limits

Memorize these: Two special trig limits…

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$

- $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$

Examples:

1) $\lim_{x \to 0} \frac{3(1-\cos x)}{x}$

2) $\lim_{x \to 0} \frac{\sin x}{\tan x}$

3) $\lim_{x \to 0} \frac{\sin x}{3x}$

4) $\lim_{x \to 0} \frac{\sin 2x}{7x}$

2.4 Notes (same day as 2.3 Day 2): Infinite Limits

Objective: Students will be able to determine if a function has an infinite limit while identifying vertical asymptotes.

Examples: Consider the following functions. Identify any vertical asymptotes, and find the requested limits.

1) $f(x) = \frac{3}{x-4}$

a) $\lim_{x \to 4^-} f(x)$

b) $\lim_{x \to 4^+} f(x)$

c) $\lim_{x \to 4} f(x)$

2) $g(x) = \frac{-3}{(x+2)^2}$

a) $\lim_{x \to -2^-} g(x)$

b) $\lim_{x \to -2^+} g(x)$

c) $\lim_{x \to -2} g(x)$

3) $h(x) = \tan x$

a) $\lim_{x \to \frac{\pi}{2}^-} h(x)$

b) $\lim_{x \to \frac{\pi}{2}^+} h(x)$

c) $\lim_{x \to \frac{\pi}{2}} h(x)$

Vertical Asymptotes: Summary
Roots of factors on the denominator that are not repeated in the numerator (where the domain is undefined.)

Examples: Use algebra to identify the vertical asymptotes for each function. Verify your conclusion with your graphing calculator.

4) $g(x) = \frac{4x^2+4x-24}{x^4-2x^3-9x^2+18x}$

5) $h(x) = \frac{-2}{\sin x}$

Properties of Infinite Limits:

Given that $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = L$

- $\lim_{x\to c} f(x) + g(x) = \infty$
- $\lim_{x\to c} f(x) \cdot g(x) = \infty, L > 0$
- $\lim_{x\to c} f(x) \cdot g(x) = -\infty, L < 0$
- $\lim_{x\to c} \frac{g(x)}{f(x)} = 0$

Example:

6) $\lim_{x\to 0} \frac{x+2}{\cot x}$

7) $\lim_{x\to 0^-} x^2 - \frac{1}{x}$

If-Time Practice: Find the requested limit analytically.

1) Find $\lim_{x\to 0}\left(x+4+\frac{(1-\cos x)}{x}\right)$.

2) Find $\lim_{x\to 0}\frac{2\sin 3x}{5x}$

3) Given $f(x) = |x-1|$
   - Write a piecewise function for $f(x)$.
   - Graph this function by analytical methods.
   - Find $\lim_{x\to 1} f(x)$

4) Draw the graph if the following conditions exist:
   - $f(x)$ if $f(-3) = -2$
   - $\lim_{x\to 0} f(x) = 5$
   - $f(0)$ is undefined
   - $\lim_{x\to 2} f(x) = f(2) = 1$

   Is your graph the only correct solution?

For #5 – 6: Use algebra to identify the vertical asymptotes for each function. Verify your conclusion with your graphing calculator.

5) $g(x) = \frac{x^2+10x-24}{9x^2-18x}$

6) $h(x) = \frac{-2}{\tan x}$

For #7 – 8: Find each limit, if possible.

7) $\lim_{x\to 0} \frac{x+1}{\tan x}$

8) $\lim_{x\to 0^-} x^2 + \frac{1}{x}$

2.6 Notes: Continuity

Objective: What is the connection between continuity, limits, and the existence of *f*(*x*)?

Definition of continuity: A function *f*(*x*) is continuous at *c* if $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = f(c)$. Thus, the following three expressions **must** be equal:

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c).$$

Exploration: For each situation below, give examples where the statement is true but the function is not continuous (draw a sketch).

1. *f*(*c*) is defined.

2. $\lim_{x \to c} f(x)$ exists.

3. $\lim_{x \to c} f(x)$ and $f(c)$ both exist

Some functions (especially rational) are **continuous on an open interval**, rather than **everywhere continuous**. There are two main types of discontinuities:

- Removable (holes)

- Non-removable (asymptotes or **jump discontinuities**)

Examples: For each function, discuss the continuity.

1) $f(x) = \frac{x-1}{x^2+x-2}$

2) $f(x) = \frac{|x+2|}{x+2}$

Examine example 2. At the point of discontinuity, does the limit exist? Would it exist from only one side?

One-sided limits:

From the right

From the left

Examples: Find each one-sided limit:

3) $\lim_{x \to 4^-} \frac{\sqrt{x-2}}{x-4}$ (No calculator!)

4) $\lim_{x \to 2^+} \begin{cases} 3x - 4 \; if \; x \le 2 \\ x^2 - 3x + 12 \; if \; x > 2 \end{cases}$

A limit exists only if $\lim_{x\to c^-} f(x) = \text{L} = \lim_{x\to c^+} f(x)$.

Example 5: Find $\lim_{x\to 3}\begin{cases} -4x+7 \; if \; x\le 3 \\ x^2-x+1 \; if \; x>3 \end{cases}$ if possible.

Reminder: Definition of Continuity:
A function is continuous at *x* = *c* on the closed interval [a, b] if it is continuous on the open interval (a, b) and $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = f(c)$.

Example 6: Use the definition of continuous to decide if $h(x)$ is continuous at *x* = 2.

6a) $h(x) = \begin{cases} -\frac{1}{2}x - e^{x-2} \; if \; x<2 \\ x^2-6 \; if \; x>2 \end{cases}$

6b) $h(x) = \begin{cases} -\frac{1}{2}x - e^{x-2} \; if \; x\le 2 \\ x^2-6 \; if \; x>2 \end{cases}$

Example 7: Find the constant *a* so that the function is continuous on the entire real line, except for *x* = 0.

$$f(x) = \begin{cases} \frac{4\sin x}{x}, if \; x < \frac{\pi}{2} \\ a-2x, if \; x \ge \frac{\pi}{2} \end{cases}$$

Intermediate Value Theorem (IVT): If *f* is continuous on the closed interval [a, b] and *L* is any number between *f*(*a*) and *f*(*b*), then there is at least one number *c* in [a, b] such that *f*(*c*) = *L*.

Example 8: Explain why the function has a zero in the given interval.

$g(x) = -4x + 3$ for the interval $\left[-\frac{5}{2}, 4\right]$

Example 9: Evaluate the following limit

$$\lim_{x\to\infty} \frac{\tan^{-1} x}{x}$$

If-time practice: For each function, discuss the continuity.

1) $f(x) = \frac{x-9}{x^2-8x-9}$

2) $f(x) = \frac{|x-2|}{x-2}$

2.5 Notes: Limits to Infinity

1) Explore the graph: $f(x) = \frac{x-3}{x-2}$

a) Identify any asymptotes.

b) $\lim_{x \to 2} f(x)$
c) $\lim_{x \to \infty} f(x)$
d) $\lim_{x \to -\infty} f(x)$

Note: $\frac{\infty}{\infty}$ is called an **indeterminate form. You can find the limit by identifying the horizontal asymptote (or by dividing each term by *x*.)**

The idea of limits at infinity is based on how quickly functions grow.

$y = \ln x$ $\qquad$ $y = x^{34}$ $\qquad$ $y = e^x$

Slowest growth ⟶ fastest growth

Limits at infinite occur at horizontal asymptotes. Review of HA for rational functions:

- Degree of numerator and denominator are the same.

- Degree of denominator is larger.

- Degree of numerator is larger.

Note: A graph can have at most 2 horizontal asymptotes… one to the right and one to the left.

Limits at Infinity:

If r is a positive rational number and c is any real number, then

- $\lim_{x \to \infty} \frac{c}{x^r} = 0$
- $\lim_{x \to -\infty} \frac{c}{x^r} = 0$

Example: Find $\lim_{x \to \infty} \left(4 + \frac{3}{x}\right)$

Examples: For each function, a) identify all asymptotes. Also, find the limits to infinity and negative infinity by using horizontal asymptotes or rates of growth.

3) $f(x) = \frac{2x}{9-x^2}$

4) $b(x) = \frac{3-2x^2}{3x-1}$

Examples: Find the following limits analytically, and then verify with graphing calc.

5) a) $\lim_{x \to \infty} \frac{x}{\sqrt{x^2+1}}$ b) $\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}}$ c) $\lim_{x \to -\infty} \frac{\sqrt{4x^2-8}}{3x}$

6) Find the limit, if possible: $\lim_{x \to \infty} \cos x$

7) Find each limit, if possible:

a) $\lim_{x \to \infty} \frac{\sqrt{9x^2+2}}{3-5x}$

b) $\lim_{x \to -\infty} \frac{\sqrt{9x^2+2}}{3-5x}$