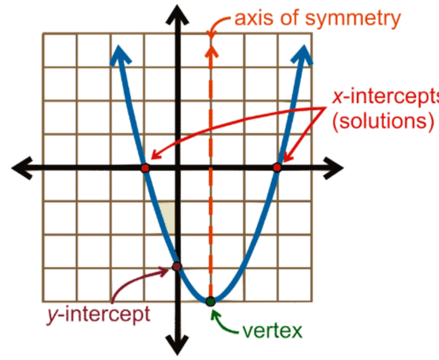
## 2.2 Notes: Quadratic Functions




**Vertex Form of a Quadratic Function: $f(x) = a(x-h)^2 + k$**

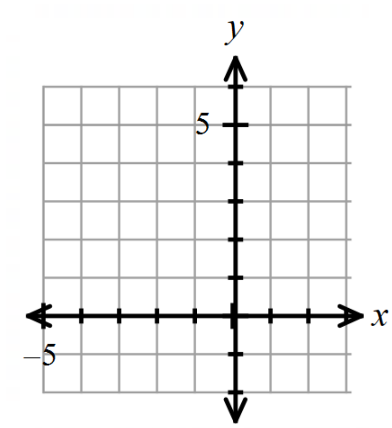
| Vertex: | Axis of symmetry: | How to find the $y$-intercept: |
|---|---|---|
| | | |

| What does $a$ tell us? | How to find $x$-intercepts: |
|---|---|
| | |

| Max or min?<br><br>Value: | Domain: | Range: |
|---|---|---|

**Example 1:** Graph $f(x) = 4(x+3)^2 - 1$ and find the requested information.

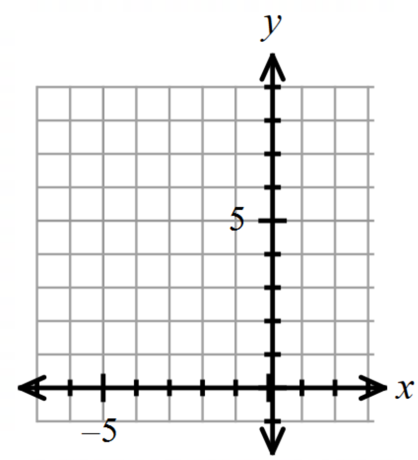
| Vertex: | Opens up or down? |
|---|---|
| Axis of Symmetry: | $x$-intercepts (if any): |
| $y$-intercept: | Max or min?<br><br>value: |
| Domain: | Range: |



| **Standard Form of a Quadratic Function: $f(x) = ax^2 + bx + c$** | | |
|---|---|---|
| Vertex: | Axis of symmetry: | $y$-intercept: |
| What does $a$ tell us? | How to find the vertex:<br><br>**Completing the Square (important skill)**<br><br>**Use** $\left(-\frac{b}{2a}, y\right)$. | How to find $x$-intercepts: |

**Example 2:** Graph $f(x) = -x^2 - 2x + 8$ and find the requested information.

| | |
|---|---|
| Vertex: | Opens up or down? |
| Axis of Symmetry: | $x$-intercepts (if any): |
| $y$-intercept: | Max or min?<br><br>Value: |
| Domain: | Range: |



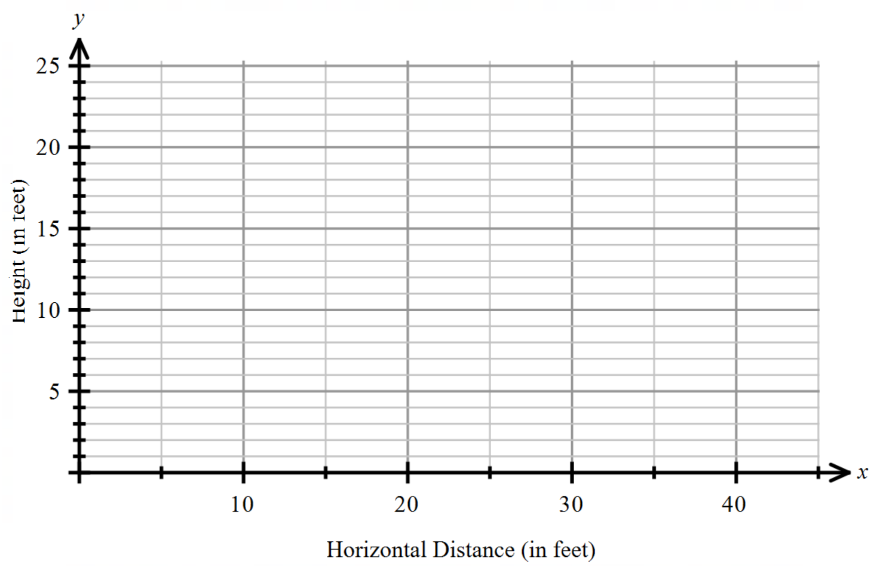
**For Examples 3 – 5, use quadratic functions to solve each problem.**

3) An archer's arrow follows a parabolic path. The height of the arrow, $f(x)$, in feet, can be modeled by $f(x) = -0.05x^2 + 2x + 5$ where $x$ is the arrow's horizontal distance in feet.

What is the maximum height of the arrow and how far from its release does this occur?

Find the horizontal distance the arrow travels before it hits the ground. (Round to the nearest foot.)

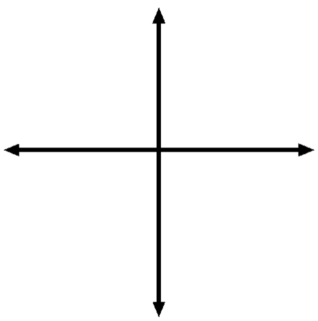
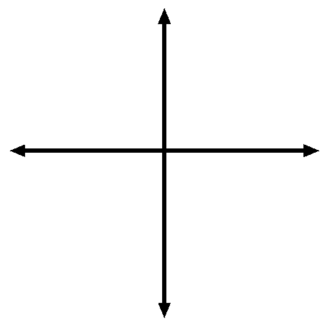
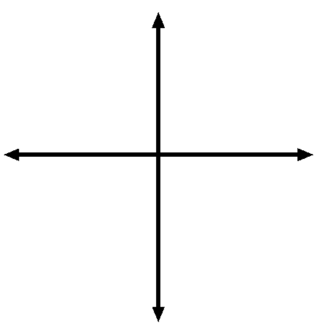
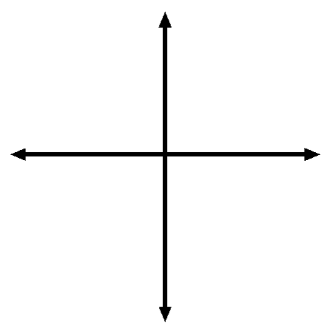
Graph the function that models the arrow's parabolic path.




4) Among all pairs of numbers whose difference is 10, find a pair whose product is as small as possible. What is the minimum product?

5) You have 100 yards of fencing to enclose a rectangular region. Find the dimensions of the rectangle that maximize the enclosed area. Also, what is the maximum area?

## 2.3 Notes: Graphs of Polynomial Functions

<table>
<tr><td colspan="2">End Behavior:</td></tr>
<tr><td colspan="2">Leading Coefficient Test:</td></tr>
<tr><td>Odd Degree with Positive Leading Coefficient<br></td><td>Odd Degree with Negative Leading Coefficient<br></td></tr>
<tr><td>Even Degree with Positive Leading Coefficient<br></td><td>Even Degree with Negative Leading Coefficient<br></td></tr>
</table>

**Example 1:** Use the leading coefficient test to determine the end behavior of the graph of each function.

a) $y = 2x^2(x-1)^2(x+5)$

b) $g(x) = -7x^5(x-3)^4(x+2)^3$

| **Zeros of a function:** | **Multiplicity of zeros:** |
|---|---|
| | |

**For Examples 2 – 3, find all zeros and their multiplicity for each function.**

2) $f(x) = x^3 + 2x^2 - 4x - 8$

3) $h(x) = -4\left(x + \frac{1}{2}\right)^2 (x - 5)^3$

**Continuous Functions:**

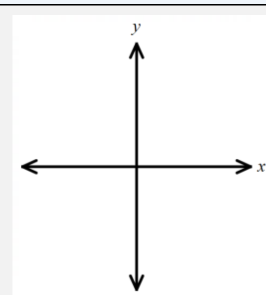
**Note:** all polynomial functions are continuous!

**Intermediate Value Theorem (IVT):**

If a function $f(x)$

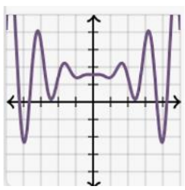
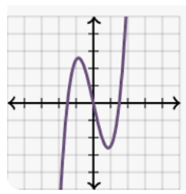
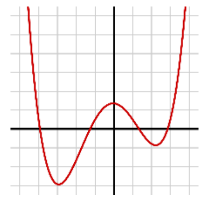
- is continuous in the interval $[a, b]$,
- and $f(a)$ and $f(b)$ have opposite signs,

then there exists at least value $c$ such that $f(c) = 0$.



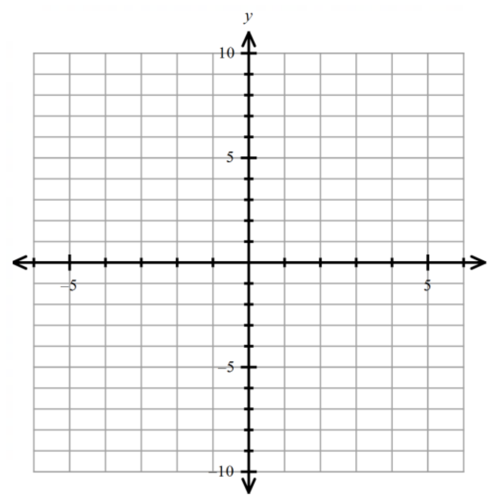

In other words, for a continuous function $f(x)$ with $f(a)$ and $f(b)$ having opposite signs, then the function $f(x)$ has at least one real zero on the interval $[a, b]$.

**Example 4)** Show that the continuous function $g(x) = 3x^3 - 10x + 9$ has a real zero on the interval $[-3, -2]$.

| Graphing Polynomial Functions | | |
|---|---|---|
| How to find $x$-intercepts (if any): | How to find the $y$-intercept: | End Behavior: |
| Types of Symmetry for Polynomial Functions | | |
| about the $y$-axis<br>(even functions)<br> | about the origin<br>(odd functions)<br> | no symmetry<br>(neither odd nor even)<br> |

**Example 5)** Graph the polynomial function and find the requested information: $y = x^4 - 2x^2 + 1$

| $x$-intercepts (if any): | $y$-intercept: |
|---|---|
| End Behavior: | |
| Symmetry, if any: | |



# 2.4 Notes: Dividing Polynomials, Remainder and Factor Theorems

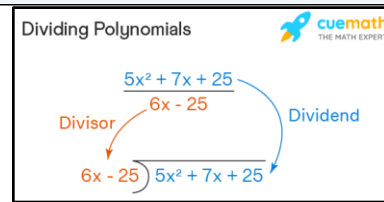
## Dividing Polynomials

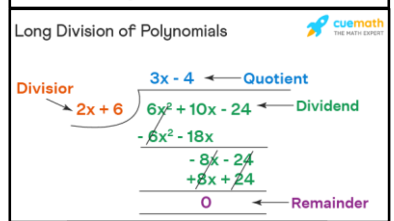
### Strategy #1: Long Division

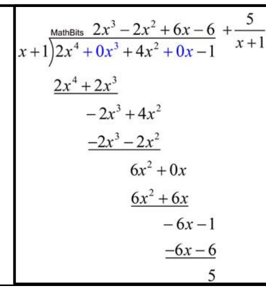
This strategy is very similar to the method of using long division to divide numbers.

**Steps:**

1. Write the dividend and divisor in descending order.
   - If needed, use 0s to represent terms of missing powers.
2. Determine what monomial you should multiply the first term of the divisor to make it identical to the first term of the dividend.
   - Hint: Write this monomial above its like term in the dividend.
3. Distribute this monomial into the divisor.
   - Write this expression below the dividend.
4. Subtract all terms from the dividend.
   - Make sure to change all the signs before subtracting.
5. Bring down *one* term from the dividend, if any have not already been subtracted from.
6. Repeat steps #2 – 5 until there are no more terms to bring down.
7. The expression left over is the remainder.
   - Write the remainder over the divisor and add it to the quotient.




**For Examples 1 – 2:** Divide by using long division.

1) Divide $y = 3x^4 - 5x^3 + 4x - 6$ by $x^2 - 3x + 5$.

2) Divide: $(x^3 + 3x^2 - 7) \div (x^2 - x - 2)$

## Dividing Polynomials

### Strategy #2: Synthetic Division

This strategy can ONLY be used if the divisor is a binomial of degree 1.

**Steps:**

1. Write the dividend and divisor in descending order.
   - If needed, use 0s to represent terms of missing powers.

2. Write the **zero** of the divisor in front of the coefficients of the dividend.
   - Be sure to use zeros as place holders for missing terms of the dividend!

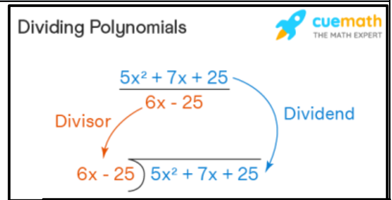
3. Bring down the leading coefficient of the dividend.

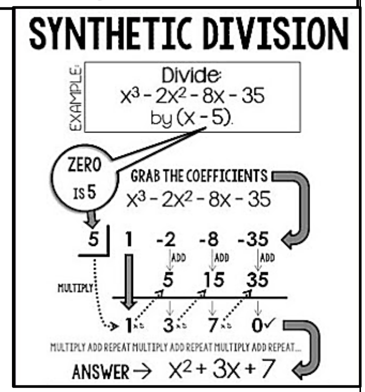
4. Multiply the zero by the number brought down.
   - Place this value underneath the next coefficient of the divisor.

5. Subtract.

6. Repeat steps #2 – 5 until there are no more terms to bring down.

7. Use the row at the bottom to write your answer.
   - The value farthest to the right is the remainder.
   - Each term increases in powers of *x* as you move to the left.




**For Examples 3 – 4: Divide by using synthetic division.**

3) Divide $2x^3 + x^2 - 8x + 5$ by $x + 3$.

4) Divide $f(x)$ by $(x - 1)$ if $f(x) = 4x^3 - 3x + 7$.

## Factor Theorem of Polynomial Functions

If $(x - c)$ is a **factor** of $f(x)$,

- then $f(x) \div (x - c)$ has a remainder of zero (0).
  - In other words, this quotient has no remainder).
- Also $c$ is a zero of $f(x)$.
- Finally, $f(c) = 0$.

**Example 5:** Consider the polynomial function $g(x) = x^4 - 4x^3 - 18x^2 + cx - 15$, where $c$ is an unknown real number. If $(x + 3)$ is a factor of this polynomial, what is the value of $c$?

| Remainder Theorem of Polynomial Functions |
|---|
| If $f(x)$ is divided by $(x - c)$,<br>• then the remainder of this quotient is the same value as $f(c)$.<br><br>(And this is why for factors of $f(x)$, $f(c) = 0$.) |

**Example 6:** Given $f(x) = 3x^3 + 4x^2 - 5x + 3$, use the Remainder Theorem to find $f(-4)$. Then use substitution to verify your answer.

**Example 7:** Solve the equation given that $x = -1$ is a zero: $15x^3 + 14x^2 - 3x - 2 = 0$

## 2.5 Notes: Zeros of Polynomial Functions

**Rational Root Theorem** (also called the Rational Zero Theorem)

Given a polynomial $f(x)$ with integer coefficients.

- Let $q$ be a *factor* of the leading coefficient of $f(x)$.
- Let $p$ be a *factor* of the constant term of $f(x)$.

Then all rational roots (or zeros) of $f(x)$ will be in the form $\frac{p}{q}$.

**To list all possible rational zeros of $f(x)$:**

- Consider all factors of $q$.
- Consider all factors of $p$.
- List all possible values of $\frac{p}{q}$.
- Make sure to include both positive and negative values for each possible zero.

**Reminder:** The number of zeros for any function of $f(x)$ is the same as the degree of $f(x)$.

- This includes both real and imaginary zeros.
- Real zeros can include both rational and irrational values.

**To find all real zeros of $f(x)$.**

- List all possible rational zeros by using $\frac{p}{q}$.
- Use synthetic division to divide $f(x)$ by some possible rational zeros.
  - Each time that you get a remainder of zero, you have found a rational zero.
- When you have a remainder of zero, use that answer as you continue testing possible rational zeros.
- Once you get a quadratic as your answer, solve this by either factoring or using the quadratic formula.

**Example 1)** List all possible rational zeros of $f(x) = 2x^3 + 7x^2 - 5x + 6$. Also, how many zero (real and imaginary) should $f(x)$ have in total?

**Example 2)** Find all zeros of $h(x) = x^3 + 7x^2 + 15x + 9$.

**Example 3)** Find all zeros of $y = x^3 + x^2 - 5x - 2$.

**Example 4:** Solve the equation for all values of $x$. $x^4 - 14x^3 + 17x^2 - 56x + 52 = 0$

**Example 5.** Write the third-degree polynomial function $f(x)$ with real coefficients such that $f(1) = 16$ and both $-3$ and $i$ are zeros.

## Descartes' Rule of Signs

Given a polynomial $f(x)$, the number of **positive real zeros** is *either*

- The same as the number of sign changes between the terms of $f(x)$.

  OR

  Less than the number of sign changes between the terms of $f(x)$ by a positive even integer.

ALSO, if $f(x)$ has only one sign change between its terms, then $f(x)$ has exactly ONE **positive** real zero.

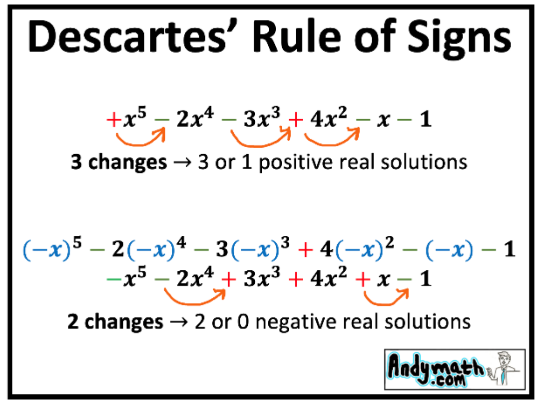
Given a polynomial $f(x)$, the number of **negative real zeros** is *either*

- The same as the number of sign changes between the terms of $f(x)$

  Or

- Less than the number of sign changes between the terms of $f(-x)$ by a positive even integer.

  ALSO, if $f(-x)$ has only one sign change between its terms, then $f(x)$ has exactly ONE **negative** real zero.




**Who is Rene Descartes?**

- 1596 – 1650
- Credited with developing the Cartesian coordinate plane (coordinate graph)
- Father of analytic geometry (points, equations of functions, etc…)
- Founder of Rationalism
- Famous argument: I think, therefore I am. "Cogito, ergo sum."

**Example 6)** Given that $f(x) = 3x^7 - 2x^5 + x^4 - 7x^2 + x - 3$. Determine the possible numbers of positive and negative real zeros of $f$.

## 2.6 Notes: Graphs of Rational Functions

What is a removable discontinuity? What is a non-removable (or infinite) discontinuity?

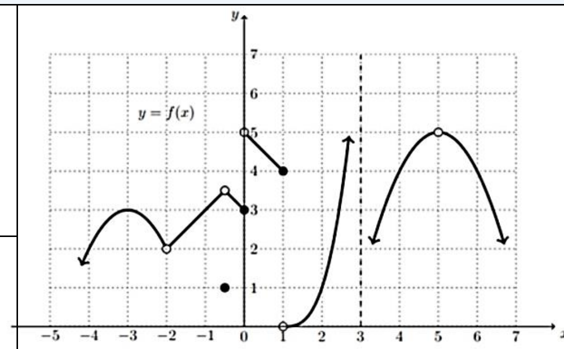
### Discontinuities

**Removable Discontinuities:**

**Non-Removable Discontinuities:**

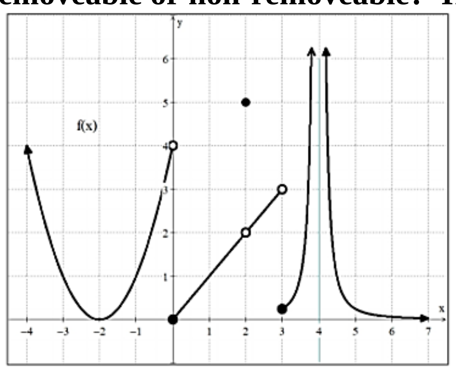
1) ___________ Asymptotes (also called Infinite Discontinuities)

2) __________ Discontinuities




**Examples 1 – 2: Identify the *x*-values for all discontinuities for each function and classify each as well (removeable or non-removeable? Hole, VA (infinite), Jump?)**

1)



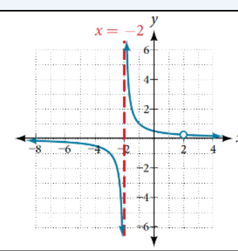

2) $f(x) = \frac{x-5}{(x+2)(x-5)}$

### Domain of Rational Functions

**Domain and Range must exclude all *x*-values of discontinuities!**

$D: \{x | x \neq x - \text{value of holes and VA}\}$

$R: \{y | y \neq y - \text{value of holes and HA}\}$




**For Examples 3 – 4, find the domain of each rational function.**

3) $\frac{x+3}{x^2-9}$

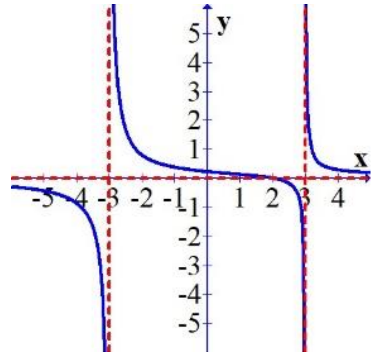
4) $\frac{x-1}{x^2+25}$

## Asymptotes of Rational Functions

**Vertical Asymptotes (VA):**

A rational function has a VA at each *x*-value that is a zero on the denominator, if that factor is not repeated on the numerator.

Form: $\boldsymbol{x = \text{number}}$

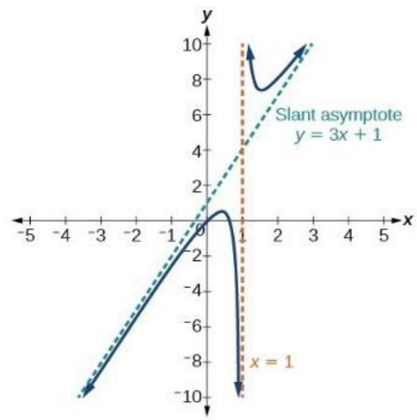



**Horizontal Asymptotes (HA):** consider the degrees of the numerator and denominator.

- If the degree is larger on the denominator, then there is a HA at $y = 0$.
- If the degree is the same on the numerator and denominator, then there is a HA at $y =$ the ratio of the leading coefficients.
- If the degree is larger on the numerator, then there is *no* HA.

Form**:** $\boldsymbol{y = \text{number}}$.

Note: a function **can cross** a horizontal asymptote, although this is rare.




**Slant Asymptote:** a rational function has a slant asymptote *only* if the degree of the numerator is ***one more*** than the degree of the denominator.

- Note: a rational function **cannot have both** a HA and a slant asymptote.
- A rational function **can have both** a VA and a slant asymptote.

To find the equation of a slant asymptote: Use synthetic or long division (ignore the remainder).

$y = \dfrac{x^2-16}{-3x+6}$ Find the slant asymptote.

$$\begin{array}{r|l} & -\tfrac{1}{3}x - \tfrac{2}{3} \\ -3x+6 & x^2 + 0x - 16 \\ & -(x^2 - 2x) \\ \hline & \quad 2x - 16 \\ & -(2x - 4) \\ \hline & \quad -12 \leftarrow \text{Remainder} \end{array}$$

$\boxed{y = -\tfrac{1}{3}x - \tfrac{2}{3}}$

**For # 5 – 7, find the equations of any asymptotes of each rational function.**

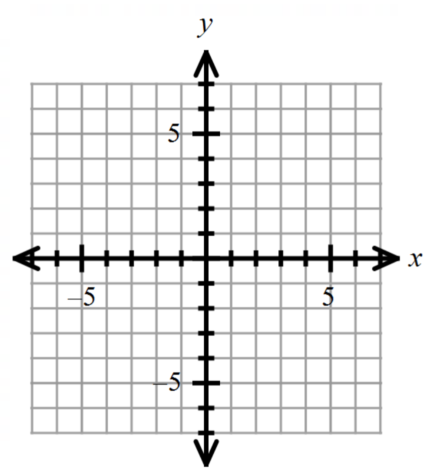
5) $\dfrac{x+3}{x^2-9}$ 6) $\dfrac{8x+7+12x^3}{5x^2-10x^3}$ 7) $\dfrac{2x^2-5x+7}{x-2}$

<table>
<tr><th colspan="2">Special Points on Rational Functions</th></tr>
<tr><td>x-intercepts (if any): The zeros of the numerator that do not repeat on the denominator.</td><td rowspan="3">Sample: $y = \frac{3x\cancel{(x+2)}(x-7)}{(x-1)\cancel{(x+2)}} = \frac{3x(x-7)}{x-1}$<br>x - int<br>VA<br>hole at $x=-2$<br>• x-int @ $x=0,\ x=7$<br>• y-int → $y = \frac{3\cdot 0(0+2)(0-7)}{(0-1)(0+2)} = \frac{0(2)(-7)}{(-1)(2)} = 0$<br>• hole at $(-2, -18)$ → $\begin{array}{c|c} x & \frac{3x(x-7)}{x-1} \\ \hline -2 & \frac{3(-2)(-2-7)}{-2-1} = \frac{54}{-3} = -18 \end{array}$</td></tr>
<tr><td>y-intercept: the value of the function when $x = 0$.</td></tr>
<tr><td>Holes:<ul><li>The x-value is the zero of a factor that repeats on the numerator and denominator.</li><li>To find the y-value of a hole, reduce out the common factor to write an equivalent expression. Evaluate this new expression at the x-value of the hole.</li></ul></td></tr>
</table>

**Example 8)** Find the $x$-intercept(s), $y$-intercept, and hole (if any) of $g(x) = \frac{x^2-4}{x^2+5x+6}$.

| Graphing Rational Functions |
|---|
| Steps:<br>1. Factor the numerator and denominator, if possible.<br>2. Identify the equations of all asymptotes. Draw them as dotted lines.<br>3. Identify the coordinates of any holes. (Reminder… used the reduced equivalent expression.) Graph these as open circles.<br>4. Use an $x-y$ table to plot at least one point on either side of each VA.<br>5. Sketch the curve to approach all asymptotes. |

**Example 9)** Graph the function $h(x) = \frac{2x^2}{x^2-9}$ and find the requested information.




VA:

HA (if any):

D:

R:

y-int:

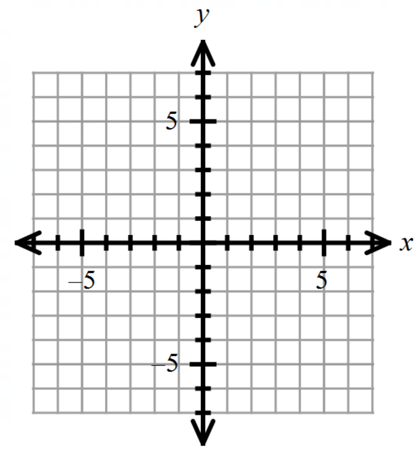
x-int (if any):

Hole (if any):

Slant Asymptote (if any):

End Behavior:

**Example 10**) Graph the function $g(x) = \frac{3x-3}{x^2-1}$ and find the requested information.




VA:

HA (if any):

D:

R:

y-int:

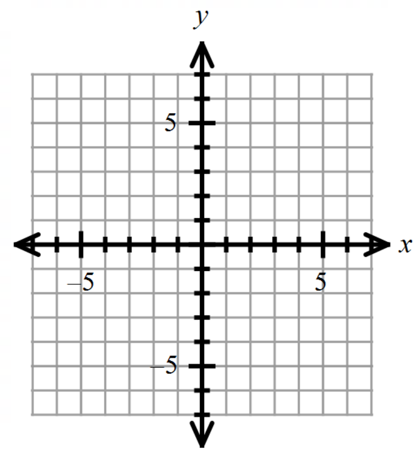
x-int (if any):

Hole (if any):

Slant Asymptote (if any):

End Behavior:

**Example 11)** Graph the function $y = \frac{x^2+4}{x-2}$ and find the requested information.




VA:

HA (if any):

D:

R:

y-int:

x-int (if any):

Hole (if any):

Slant Asymptote (if any):

End Behavior:

# 2.6 Day 2 and 2.7 Notes: Graphing Rational Functions; Polynomial and Rational Inequalities

| 2.6, continued: Graphing Form of a Rational Function | |
|---|---|
| **Graphing Form:** $y = \frac{a}{x-h} + k$<br><br>*VA at $x = h$<br>*HA at $y = k$<br><br>Note: rational functions in this form do not have holes or slant-asymptotes. | Steps:<br>1. Identify the equations of all asymptotes. Draw them as dotted lines.<br>2. Use an $x - y$ table to plot at least one point on either side of each VA.<br>Sketch the curve to approach all asymptotes. |

**Example 1)** Graph the function $g(x) = \frac{1}{x+2} - 1$ and find the requested information.

VA:

HA (if any):

D:

R:

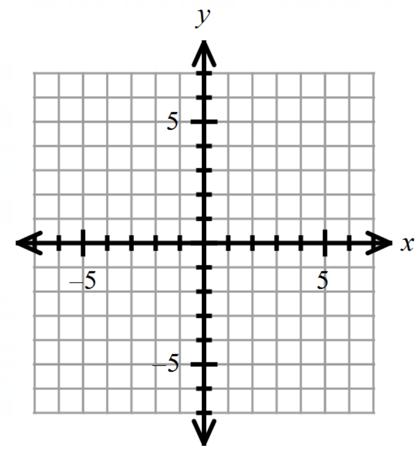
y-int:

x-int (if any):

Hole (if any):

Slant Asymptote (if any):

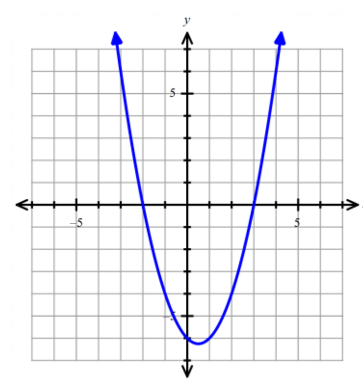
End Behavior:




## 2.7 Exploration: Consider $f(x) = (x+2)(x-3)$.

- For what interval(s) of $x$ is $f(x) \leq 0$?

- For what interval(s) of $x$ $f(x) > -4$?



## 2.7: Solving Polynomial and Rational Inequalities

Steps:

1. For polynomials, get a 0 on one side of the inequality.
2. Find all values of *x* where there is a zero or a discontinuity.
    - In other words, find the zeros of the numerator and denominator.
3. Set up a table or number line with intervals based on the values of *x* found in step 2.
4. Test values of *x* within each interval to see if they satisfy the inequality or not.
5. Use your table to find the interval(s) for the solution.

**Sample:** Solve $\frac{x^2-2x-15}{x-2} \geq 0$.

$$\frac{(x-5)(x+3)}{x-2} \geq 0$$

$-3 \quad 2 \quad 5$

$x = 5, -3$ and $x \neq 2$

| $(-\infty, -3)$ | $(-3, 2)$ | $(2, 5)$ | $(5, \infty)$ |
|---|---|---|---|
| Test $x=-4$ | Test $x=0$ | Test $x=3$ | Test $=6$ |
| $\frac{(-4-5)(-4+3)}{(-4-2)}$ | $\frac{(0-5)(0+3)}{(0-2)}$ | $\frac{(3-5)(3+3)}{(3-2)}$ | $\frac{(6-5)(6+3)}{(6-2)}$ |
| negative $\geq 0$ NO | positive $\geq 0$ YES | negative $\geq 0$ NO | pos $\geq 0$ YES |

$-3 \quad 2 \quad 5$

$[-3, 2)$ and $[5, \infty)$

**Examples 2 – 3: Solve each inequality, and graph the solution set on the real number line.**

2) $2x^2 + x > 15$

3) $x^3 + x^2 \leq 4x + 4$

**Examples 4 – 5: Solve each inequality, and graph the solution set on the real number line.**

4) $\frac{x+1}{x+3} \geq 2$

5) $(2-x)^2\left(x-\frac{7}{2}\right) < 0$

See next page for optional additional practice for rational functions:

**Extra *optional* practice with rational functions:**

1) Graph the function $g(x) = \frac{x^2-5x}{x^2-25}$ and find the requested information.

VA:

HA (if any):

D:

R:

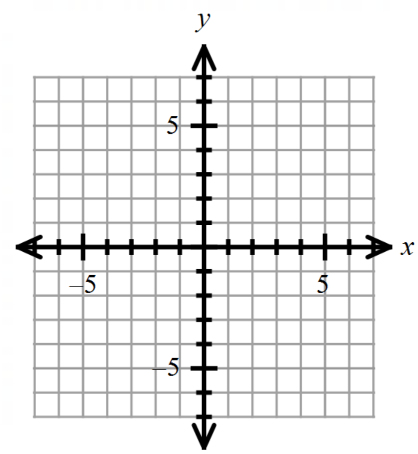
$y$-int:

$x$-int (if any):

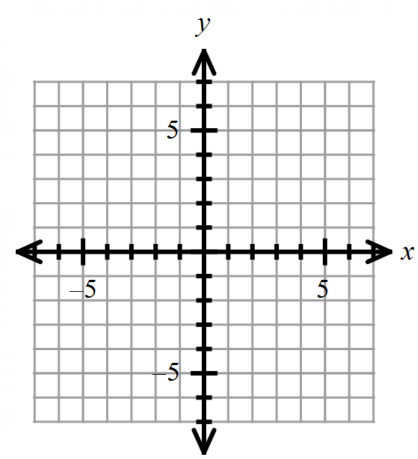
Hole (if any):

Slant Asymptote (if any):

End Behavior:




2) Consider $g(x) = \frac{2x^2-5x+7}{x-2}$ Find the equations for all asymptotes and sketch the rational function.




3) Consider $g(x) = \frac{-4+x^2}{3x^2+1}$ Find the equations for all asymptotes, as well as the coordinates for all intercepts.