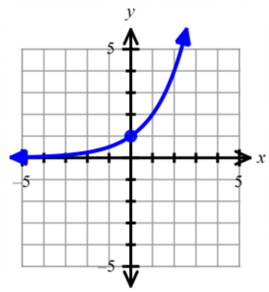
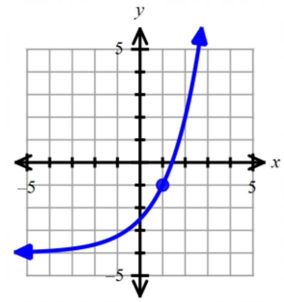
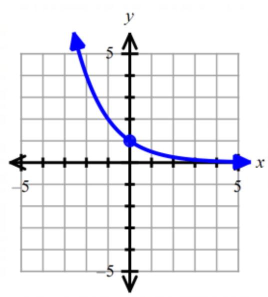
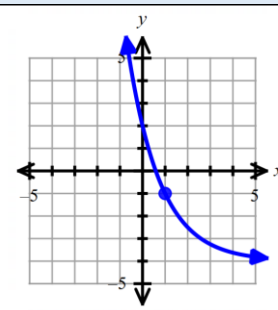
## 3.1 Notes: Graphs of Exponential and Logarithmic Functions

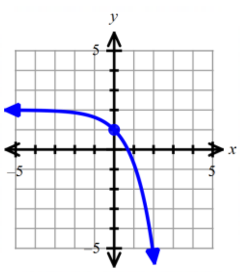
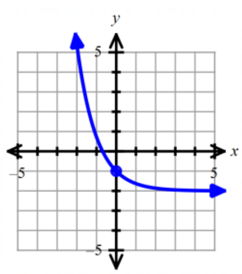
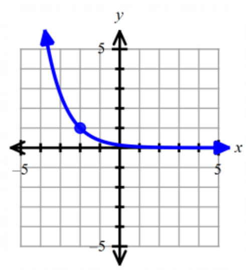
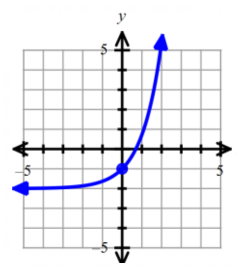
### Parent Functions for Exponential Functions

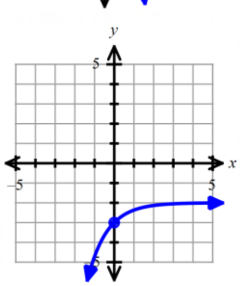
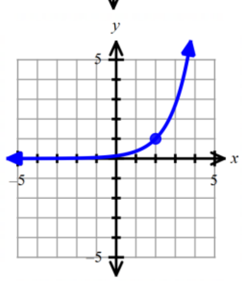
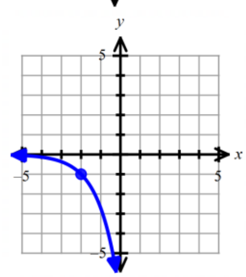
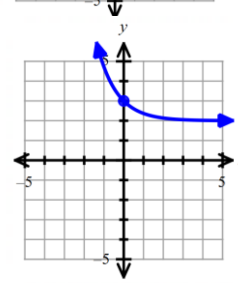
| | $y = b^x$ where $b > 1$ | $y = a \cdot b^{x-h} + k$ |
|---|---|---|
| **Exponential Growth Functions** | HA: $y = 0$<br>Anchor point: $(0, 1)$<br>As $x \to \infty, y \to$ _____<br>Domain:<br>Range:<br>*A table of values can also be used.<br><br>Sample: $y = 2^x$ | HA: $y = k$<br>Anchor point: $(h, a + k)$<br>As $x \to \infty, y \to$ _____<br>Domain:<br>Range:<br><br>Sample: $y = 3 \cdot 2^{x-1} - 4$ |
| | $y = b^x$ where $0 < b < 1$ | $y = a \cdot b^{x-h} + k$ |
| **Exponential Decay Functions** | HA: $y = 0$<br>Anchor point: $(0, 1)$<br>As $x \to \infty, y \to$ _____<br>Domain:<br>Range:<br>*A table of values can also be used.<br><br>Sample: $y = \left(\frac{1}{2}\right)^x$ | HA: $y = k$<br>Anchor point: $(h, a + k)$<br>As $x \to \infty, y \to$ _____<br>Domain:<br>Range:<br><br>Sample: $y = 3 \cdot \left(\frac{1}{2}\right)^{x-1} - 4$ |

**Examples 1 – 8: Match each equation with its graph below. No graphing calculators allowed.**

1) $y = e^x - 2$ 2) $y = e^{x-2}$ 3) $y = -e^x + 2$ 4) $y = -e^{x+2}$

5) $y = \left(\frac{1}{e}\right)^x - 2$ 6) $y = e^{-x} + 2$ 7) $y = \left(\frac{1}{e}\right)^{x+2}$ 8) $y = -e^{-x} - 2$

A) 
B) 
C) 
D) 

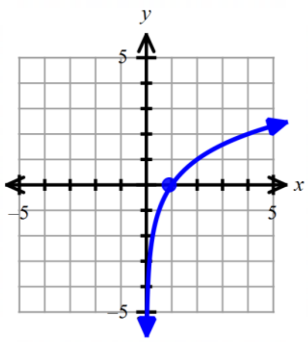
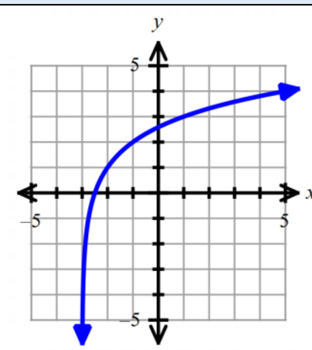
E) 
F) 
G) 
H) 

9) What is the domain and range (in interval notation) for graph E above?

10) What is the domain and range (in interval notation) for graph B above?

**Transformations from $y = e^x$ to $y = a \cdot b^{x-h} + k$**

11) Describe the transformations from $y = e^x$ to $y = -3e^{x+1} + 5$.

<table>
<tr><th colspan="2">Parent Functions for Logarithmic Functions</th></tr>
<tr><th>$y = \log_b x$ where $b > 1$</th><th>$y = \log_b(x - h) + k$ where $b > 1$</th></tr>
<tr><td>VA: $x = 0$<br><br>Anchor point: (1, 0)<br><br>As $x \to \infty, y \to$ _____<br><br>Domain:<br><br>Range:<br><br><br>*A table of values can also be used.<br><br>Sample: $y = \log_2 x$</td><td>VA: $y = h$<br><br>Anchor point: $(h + 1, k)$<br><br>As $x \to \infty, y \to$ _____<br><br>Domain:<br><br>Range:<br><br><br>*A table of values can also be used.<br><br>Sample: $y = \log_2(x + 3) + 1$</td></tr>
</table>

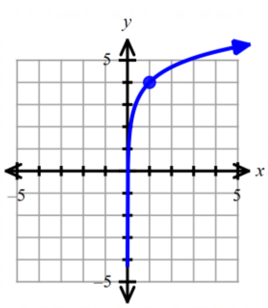
**Examples 12 – 15: Match each equation with its graph without using a graphing calculator.**

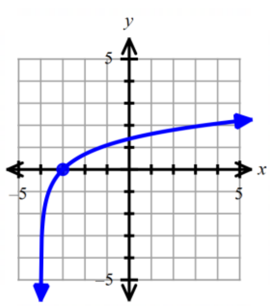
12) $y = \ln(x + 4)$

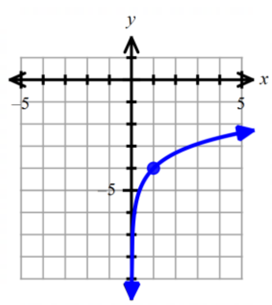
13) $y = \ln(x - 4)$

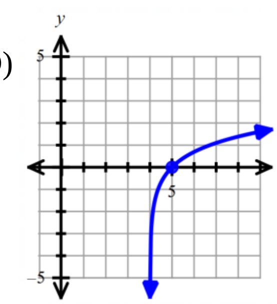
14) $y = \ln x + 4$

15) $y = \ln x - 4$

A) 

B) 

C) 

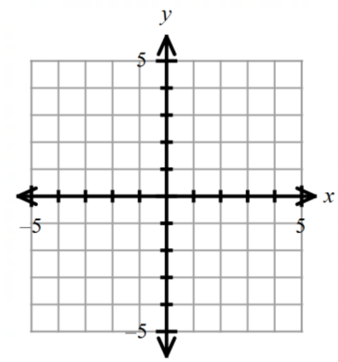
D) 

16) What is the domain and range (in interval notation) for graph B above?

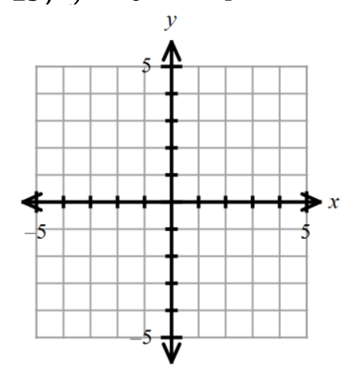
17) What is the domain and range (in interval notation) for graph C above?

**For #18 – 20:** Sketch each exponential function without a calculator. Include the HA and anchor point.

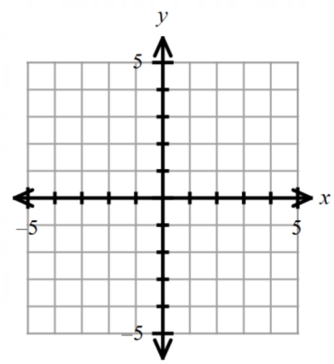
18) $y = e^x + 3$



19) $y = e^{-x} - 4$



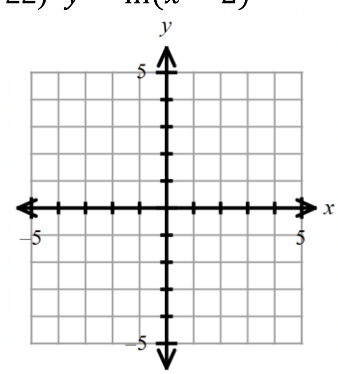
20) $y = -e^{x+1} + 2$



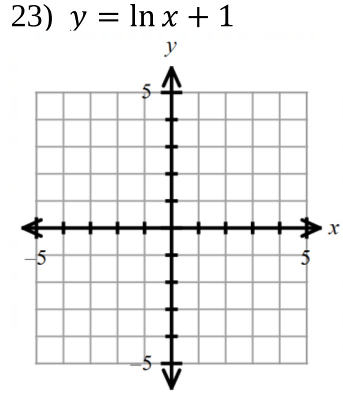
21) What is the domain and range of #20? What are the transformations from $y = e^x$?

**For #22 – 24:** Sketch each logarithmic function without a calculator. Include the VA and anchor point.

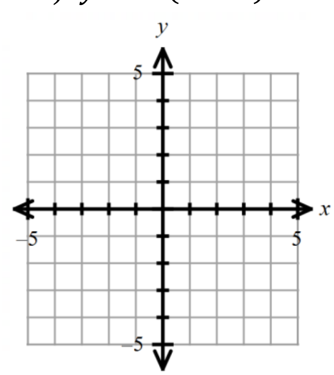
22) $y = \ln(x - 2)$



23) $y = \ln x + 1$



24) $y = \ln(x + 3) - 2$

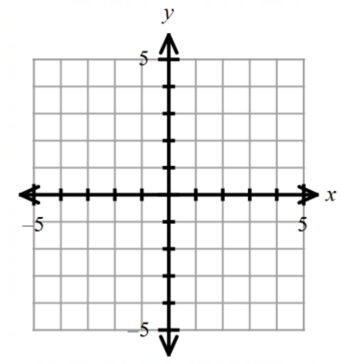


25) What is the domain and range of #22?

26) What are the transformations from $y = \ln x$ to the equation from #24?

## 3.2 and 3.3 Notes: Properties of Logarithmic and Exponential Expressions

**Exploration:** Use a graphing calculator to graph $f(x) = \ln x$ and $g(x) = e^x$. Draw a sketch on the provided coordinate systems.




Find the domain and range of each function. What do you notice?

Compare and contrast $f(x)$ and $g(x)$. Make as many observations as you can.

| Logarithmic and Exponential Equations and Expressions | |
|---|---|
| Logarithmic and Exponential functions are ____________________ of each other,<br>as long as they use the _________________________ base. | |
| **Parts of Logarithmic and Exponential Expressions** | $\boldsymbol{\log_b a = x}$ and $\boldsymbol{b^x = a}$.<br>**Base:**<br>**Argument (of log):**<br>**Exponent:** |
| **Converting Equations to Logarithmic or Exponential Form** | $\boldsymbol{\log_b a = x \longrightarrow b^x = a}$. |
| **Inverse Properties** with Logs and Exponentials of the Same Base | $\boldsymbol{\log_b b^x = x}$ and $\boldsymbol{b^{\log_b x} = x}$ |

**For #1 – 3: Re-write each equation in logarithmic or exponential form.**

1) $\log_x 81 = 4$ 2) $\log a = x$ 3) $2^3 = x$

**For #4 – 13: Simplify each expression without a calculator.**

4) $\log_3 81$

5) $\log_5 \sqrt[7]{5}$

6) $\log_{1/4} 256$

7) $\ln e^5$

8) $\log 1$

9) $\log_{36} 6$

10) $\log_3 27^x$

11) $\log_5 25^{-2x}$

12) $\ln e^{8-x}$

13) $\log_3 27 + \ln 1 - \log 10^4 - 5\ln e^4 - \log_2 32 + e^{\ln 6}$

**Note:** important values you ***must know:*** $\ln e =$ ____; $\ln 1 =$ _____; $\ln 0 =$ _____

| Compound Interest Formulas | |
|---|---|
| **Compounded *n* times per year** | **Compounded continuously** |
| $A = P\left(1 + \frac{r}{n}\right)^{nt}$ | $A = Pe^{rt}$ |

14) A person wants to invest \$3,000 in a saving account for 4 years. The bank has two options. The first option compounds interest weekly at a rate of 5.4%. The second option compounds interest continuously at a rate of 5%. Which option should you choose? Explain your choice.

| Properties of Logarithmic Expressions | | | |
|---|---|---|---|
| **Product Property** | $\log_b(m \cdot n) = \log_b m + \log_b n$ | **Power Property** | $\log_b(m^n) = n \cdot \log_b m$ |
| **Quotient Property** | $\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$ | **Change - of - Base Formula** | $\log_b m = \frac{\ln m}{\ln b}$<br>$\log_b m = \frac{\log_a m}{\log_a b}$ |
| **Condensing Log Expressions** | 1. Move coefficients to powers of the argument.<br>2. Change addition to multiplication of the arguments.<br>3. Change subtraction to multiplication of the arguments. | **Expanding Log Expressions** | 1. Change multiplication of the argument to addition of two logs of the same base.<br>2. Change division of the argument to subtraction of two logs of the same base.<br>3. Move any powers of the argument to coefficients of the log. |

**Examples 15 – 19: Condense each logarithmic expression.**

15) $\ln 4 + 3\ln 3 - \ln 12$

16) $\frac{1}{3}[2\ln(x+5) - \ln x - \ln(x^2-4)]$

17) $\log_2(6x+1) - \frac{1}{2}\log_2 2y + \log_2 24 - \log_2 3z$

18) $-\frac{1}{2}\ln 16 + 2\ln 3$

19) $\ln 5 + \frac{1}{3}\ln(a+2) - 2\ln b - \ln c$

**For #20 – 22, expand each logarithmic expression.**

20) $\log_7 \frac{3\sqrt[4]{x}}{5y^3}$

21) $\ln \frac{2x^3y}{7z^4}$

22) $\log \frac{7ab^4}{(c+d)^5}$

**For #23 – 24:** Use the change-of-base formula to evaluate each logarithm. Give an exact solution and an approximate solution to 3 decimals.

23) $\log_5 8$

24) $\log_8 14$

## 3.4 Notes: Solving Exponential and Log Equations

| Solving Exponential Equations | | | |
|---|---|---|---|
| **Option 1:**<br><br>**Both sides can be written with the same base.** | 1. Write both sides with the same base.<br>2. Set the exponents equal and solve. | **Option 2:**<br><br>**Both sides cannot be written with the same base.** | 1. If possible, isolate the base and exponent term.<br>a. Use inverse operations<br>or<br>b. Factor the expression<br>2. Take the ln (or another log) to both sides.<br>3. Use properties of logs to solve. |

**For #1 – 8:** Solve each exponential equation for the variable. If needed, round to one decimal place.

1) $e^{x+6} = e^{3x}$

2) $9^{2x} = 27^{x-1}$

3) $\frac{1}{343} = 7^{2x+5}$

4) $2^{x-4} = 5.3$

5) $5 \cdot 3^{x-5} + 1 = 21$

6) $3^{x-4} = 2^{3x+1}$

7) $e^{2x} - 8e^{x} + 7 = 0$

8) $5e^{2x} - 5e^{x} = 0$

<table>
<tr><th colspan="4">Solving Logarithmic Equations</th></tr>
<tr>
<td>Only one logarithmic term</td>
<td>1. Isolate the log term, if possible.<br>2. Convert the equation to an exponential expression (or exponentialize each side)<br>3. Solve.<br><br>Sample: $\ln(5x) + 6 = 8$<br>$\ln(5x) = 2$<br>$e^2 = 5x$<br>$\boxed{\frac{e^2}{5} = x}$<br>OR<br>$\cancel{e}^{\cancel{\ln}(5x)} = e^2$<br>$5x = e^2$<br>$\boxed{x = \frac{e^2}{5}}$</td>
<td>One log term on each side (with the same base)</td>
<td>1. Note: this strategy only works if there are no additional terms/constants in the equation.<br>2. Set the arguments equal and solve.<br><br>Sample: $\ln(3x - 2) = \ln(4 - x)$<br>$3x - 2 = 4 - x$<br>$4x = 6$<br>$\boxed{x = \frac{3}{2}}$</td>
</tr>
<tr>
<td>More than two logarithmic terms on one side.</td>
<td colspan="3">1. Condense to one log term on each side.<br>2. Move any powers of the argument to coefficients of the log.<br>3. Use the row above to find an appropriate strategy to solve.<br><br>Samples: $\ln x + \ln(x - 2) = \ln 8$<br>$\ln x(x-2) = \ln 8$<br>$x^2 - 2x = 8$<br>$x^2 - 2x - 8 = 0$<br>$(x-4)(x+2) = 0$<br>$\boxed{x = 4}$; $\cancel{x = -2}$<br><br>$\log_4 x + \log_4(x + 6) = 2$<br>$\log_4 x(x+6) = 2$<br>$\log_4(x^2 + 6x) = 2$<br>$4^2 = x^2 + 6x$<br>$x^2 + 6x = 16$<br>$x^2 + 6x - 16 = 0$<br>$(x+8)(x-2) = 0$<br>$\cancel{x = -8}$ $\boxed{x = 2}$</td>
</tr>
<tr>
<td>Extraneous Solutions</td>
<td colspan="3">All arguments for all log expressions must be positive values. Reject any values that would make the argument a negative value or zero.</td>
</tr>
</table>

**For #9 – 12:** Solve each logarithmic equation and check for extraneous solutions

9) $\log_5(4x - 7) = \log_5(x + 5)$

10) $\log_4(5x - 1) = 3$

11) $\log_4(x + 12) + \log_4 x = 3$

12) $\log x + \log(x + 5) = \log 24$

**More Examples: Solve each problem. Round your final answer to one decimal place.**

13) The population of deer in a forest preserve can be modeled by the equation $P(t) = 50 + 200\ln(t + 1)$, where $t$ is the time in years from the present. In how many years will be deer population reach 500?

14) How long would you have to invest $30,000 in an account earning 6% interest compounded continuously so that you have a total of $40,000?

When solving equations with logs and exponential functions, if you choose to write your intermediate values (non-final answers) as decimals, **use at least 6 decimal places** for accuracy. Alternatively, write values as fractions.

15) Sally invests $350 in an account earning 5% interest, compounded annually. How long will it take her to earn $50 in interest?

## 3.5 Exponential Growth and Decay

| **Solving Exponential Growth/Decay Equations** | | | |
|---|---|---|---|
| **Basic Formula** | $A = A_0 e^{kt}$<br><br>Growth: $k > 0$ Decay: $k < 0$<br><br>Note: Half-life problems are exponential decay problems. | **Hints and Strategies** | 1. Write ordered pairs in the form $(t, A)$ … (time, amount)<br>2. Use substitution with one ordered pair to solve for $k$. Either use an exact answer, or include 6 decimal places.<br>3. Re-write the customized equation (with the known value for $k$).<br>4. Use substitution with another ordered pair to solve the problem. |

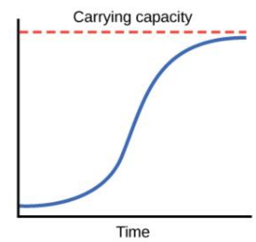
1) In 1990, the population of Africa was 643 million, and by 2006 it had grown to 906 million.

a) Use the exponential growth model, where $t$ is the number of years after 1990, to find the exponential growth function that models the data.

b) In which year will Africa's population reach 2100 million (2.1 billion)?

2) Strontium-90 is a waste product from nuclear reactors. The half-life of strontium-90 is 28 years, meaning that after 28 years a given amount of the substance will have decayed to half the original amount. Find the exponential decay model for strontium-90.

Then suppose that a nuclear accident occurs and releases 60 grams of strontium-90 into the atmosphere. How many years (rounded to one decimal place) will it take for strontium-90 to decay to a level of 10 grams?

| Logistic Growth Models | | | |
|---|---|---|---|
| **Formula** | $A = \dfrac{c}{1 + ae^{-bt}}$<br><br>$a, b,$ and $c$ are constants with $c > 0$ and $b > 0$. | **Graph and Features** | • input is $t$ (often "time")<br>• $c$ is a horizontal asymptote<br>• $c$ is called the "carrying capacity" or "limiting amount," and it is the maximum height (or amount) of the function.<br> |

3) In a learning theory project, psychologists discovered that $f(t) = \frac{0.8}{1+e^{-0.2t}}$ is a model for describing the proportion of correct responses after $t$ learning trials.

a) Find the proportion of correct responses prior to learning trials taking place $(t = 0)$

b) Find the proportion of correct responses after 10 learning trials

c. What is the limiting size of $f(t)$, the proportion of correct responses, as continued learning trials take place?

| Newton's Law of Cooling | | | |
|---|---|---|---|
| **Formula and variables** | $T = C + (T_0 - C)e^{-kt}$<br><br>$T =$<br><br>$C =$<br><br>$T_0 =$<br><br>$k =$<br><br>$t =$ | **When is this used?** | • CSI investigators and coroners use Newton's Law of Cooling to determine the time of death for a deceased person.<br><br>• Newton's Law of Cooling is used in business, industry, medicine, etc… |

**Newton's Law of Cooling:** $T = C + (T_0 - C)e^{-kt}$

4) An object is heated to 100°C. It is left to cool in a room that has a temperature of 30°C. After 5 minutes, the temperature of the object is 80°C.

a) Use Newton's Law of Cooling to find a model for the temperature of the object, $T$, after $t$ minutes.

b) What is the temperature of the object after 20 minutes? Round your answer to one decimal place.

c) When will the temperature of the object be 35°C? Round your answer to one decimal place.

5) Rewrite $y = 4(7.8)^x$ in terms of base $e$. Express the answer in terms of natural logarithm and then round to three decimal places. (Hint: $b^{\log_b x} = x$)